MAT360 Section Summary: 3.3
Osculating polynomial interpolation

1. Summary

We may want to fit more than function values: we may also have derivative (first, or even higher) that we want to fit as well. In that case, we may need to require higher degree polynomials that incorporate the derivative information.

The easiest way to derive the interpolator is by modifying Newton's form of the interpolating polynomial just a little.

2. Definitions

- **Osculating polynomial**: Let $x_0, \ldots, x_n$ be $n+1$ distinct points in $[a, b]$, and m_i be a non-negative integer associated with x_i. Suppose that $f \in C^m[a, b]$, where m is the maximum of the m_i. Then the **osculating polynomial** approximating f is the polynomial $P(x)$ of least degree such that

$$P^{(k)}(x_i) = f^{(k)}(x_i)$$

for each $i = 0, \ldots, n$ and $k = 0, \ldots, m_i$.

- **Hermite polynomials**: those osculating polynomials for which $m_i = 1$ for all i (that is, we fit the points and the slopes − no higher derivatives).

3. Properties/Tricks/Hints/Etc.

The trick to computing Hermite polynomials is to "trick" the Newton interpolating polynomial into incorporating the true derivative information. As you no doubt recall, the derivative information is contained in the divided differences:

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_1, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}$$

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where

$$f[x_0] = f(x_0)$$

For each derivative in osculating interpolation that we intend to include, we simply include redundant point information. To fit a cubic, for example, and the two end points with point and derivative information, we would construct the following table:

$$
\begin{array}{llll}
x_0 & f[x_0] & & \\
 & & f[x_0, x_0] \equiv f'(x_0) & \\
x_0 & f[x_0] & & f[x_0, x_0, x_1] \\
 & & f[x_0, x_1] & & f[x_0, x_0, x_1, x_1] \\
x_1 & f[x_1] & & f[x_0, x_1, x_1] \\
 & & f[x_1, x_1] \equiv f'(x_1) & \\
x_1 & f[x_1] & &
\end{array}
$$

Then the interpolating polynomial might be, for example,

$$P_3(x) = f[x_0] + f[x_0, x_0](x-x_0) + f[x_0, x_0, x_1](x-x_0)^2 + f[x_0, x_0, x_1, x_1](x-x_0)^2(x-x_1)$$

Example: #9 asks us to verify that this polynomial meets the requirements of the osculating cubic....

Check that P_3 fits function values + slopes.

$P_3(x_0) = f[x_0] = f(x_0)$ ✓ $P_3'(x_0) = f[x_0, x_0] = f'(x_0)$ ✓

$P_3(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + \left[\dfrac{f[x_0, x_1] - f'(x_0)}{x_1 - x_0} \right](x_1 - x_0)^2$

$= f(x_0) + f'(x_0)(x_1 - x_0) + f[x_0, x_1](x_1 - x_0) - f'(x_0)(x_1 - x_0)$

$= f(x_0) + \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}(x_1 - x_0) = f(x_1)$ ✓

The most important case we'll consider is one of osculating splines, generally cubic splines, that fit pairs of points and the derivatives at those points. Fitting two things on two ends gives four constraints,

and that leads to a complete determination of the four parameters of a cubic.

Example: 1a:

x	$f(x)$	$f'(x)$
8.3	17.56492	3.116256
8.6	18.50515	3.151762

The error information is included in the usual way for Hermite interpolation: it is estimated using the same formula as in theorem 3.3:

$$\overline{f(x) = P(x) + \frac{f^{(2n+2)}(\xi(x))}{(2n+2)!}(x-x_0)^2 \cdots (x-x_n)^2}$$

Example: #2a, and bound the error for the interval $[8.3, 8.6]$.

To incorporate m_i derivatives at a certain point x_i, one makes m_i entries for x_i in the table, and works the derivative information up the divided difference table. Simple!

Example: include the following second derivative information into the previous function:

x	$f(x)$	$f'(x)$	$f''(x)$
8.3	17.56492	3.116256	0.1204819
8.6	18.50515	3.151762	0.1162791

$x_0 \quad f(x_0)$

$\qquad\qquad f[x_0, x_0] = f'(x_0)$

$x_0 \quad f(x_0) \qquad\qquad\qquad\qquad f''(x_0)$

$\qquad\qquad f'(x_0) \qquad\qquad\qquad\qquad f[x_0, x_0, x_1, x_1]$

$x_0 \quad f(x_0) \qquad\qquad\qquad f[x_0, x_0, x_1] \qquad\qquad\qquad\qquad f[x_0, x_0, x_0, x_1, x_1]$

$\qquad\qquad f[x_0, x_1] \qquad\qquad\qquad\qquad f[x_0, x_0, x_1, x_1]$

$x_1 \quad f(x_1) \qquad\qquad\qquad f[x_0, x_1, x_1] \qquad f[x_0, x_0, x_1, x_1, x_1]$

$\qquad\qquad f'(x_1) \qquad\qquad\qquad\qquad f[x_0, x_1, x_1, x_1]$

$x_1 \quad f(x_1) \qquad\qquad\qquad f''(x_1)$

$\qquad\qquad f'(x_1)$

$x_1 \quad f(x_1)$

And the climactic

$$f[x_0, x_0, x_0, x_1, x_1, x_1]$$

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