

MAT360 Section Summary: 4.2 Richardson's Extrapolation

1. Summary

This is variation of the process we used for the creation of the three- and five-point methods for differentiation, which we created from Taylor series and points with fixed step sizes.

We're looking at methods that have formulas of the form

$$N(h) = M + K_1 h + K_2 h^2 + K_3 h^3 + K_4 h^4 + \dots$$

and we want to use several "attacks", with various stepsizes, to eliminate the terms of lower order in h . These methods and the solutions to this problem are all best expressed in the language of linear algebra: the only change is in the coefficients. You'll see that in Richardson's extrapolation, we don't rely on evenly spaced points, but rather on points that are getting smaller by powers of 2. No big deal!

It's important to realize that we can use the same procedure with any group of points: consider one of the distances as " h ", and all other distances can be expressed in h -units. Then we build our linear algebra tables, and away we go.

2. Properties/Tricks/Hints/Etc.

Let's start with an easy one, which works out to be equally spaced points, as well. Use the forward difference approximation to the derivative and steps of h and $h/2$ to get a better approximation.

$$\overline{f(x_0 + h)} = f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{3!} f'''(\xi(x_0))$$

$$f(x_0 + \frac{h}{2}) = f(x_0) + \frac{h}{2} f'(x_0) + \frac{h^2}{2^2 \cdot 2} f''(x_0) + \frac{h^3}{2^3 \cdot 6} f'''(\psi(x_0))$$

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} = \frac{h}{2} f'(x_0) + \frac{h^2}{3!} f'''(\xi)$$

$$f'(x_0) = \underbrace{\frac{f(x_0 + \frac{h}{2}) - f(x_0)}{\frac{h}{2}}}_{N(\frac{h}{2})} = \frac{h}{2^2} f''(x_0) + \frac{h^2}{2^3 \cdot 6} f'''(\psi)$$

$$2N(\frac{h}{2}) - N(h) = f'(x_0) + O(h^2)$$

Better order using a pair
Better method!

Question: what does this work out to? Does this look familiar?

Now how do we generalize this for more points? Consider eliminating up to (but not including) fourth order in h , $O(h^4)$:

$$\begin{bmatrix} \text{coefficient} & M & k_1 h & k_2 h^2 & k_3 h^3 \\ h & 1 & 1 & 1 & 1 \\ h/2 & 1 & 1/2 & 1/4 & 1/8 \\ h/4 & 1 & 1/4 & 1/16 & 1/64 \\ h/8 & 1 & 1/8 & 1/64 & 1/512 \end{bmatrix}$$

We could simply provide this to a linear algebra solver, and ask it for the coefficients of the four methods that produces

$$\begin{bmatrix} \text{coefficient} & M & k_1 h & k_2 h^2 & k_3 h^3 \\ & 1 & 0 & 0 & 0 \end{bmatrix}$$

We have four equations in the four unknown weights of the four methods. One way of solving would be to use the strategy of the extrapolation table (e.g. Table 4.5, p. 181): each successive method eliminates one additional power of h .

$$C_1 = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{\text{mult } (2)} A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1/2 & 1/4 & 1/8 \\ 1 & 1/4 & 1/16 & 1/64 \\ 1 & 1/8 & 1/64 & 1/512 \end{bmatrix}$$

Next sweep (next column in table):

$$C_2 = \frac{1}{3} \begin{bmatrix} -1 & 4 & 0 \\ 0 & -1 & 4 \end{bmatrix} \xrightarrow{\text{mult } (2)} C_1, A$$

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Final sweep:

$$C_3 = \frac{1}{7} \begin{bmatrix} -1 & 8 \end{bmatrix} \xrightarrow{\text{mult } (2)} C_2, A$$

and equals $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$