# MAT360 Section Summary: 4.3

Elements of Numerical Integration (part I)

## 1. Summary

The Newton-Cotes formulas (open and closed) that are derived in this section allow us to approximate an integral by a weighted sum of discrete points on $[a,b]$, the interval of interest:

$$I = \int_a^b f(x)dx \approx \sum_{i=0}^{n} a_i f(x_i)$$

In this section we focus on a single elemental "unit", using just two or three points; in the next section we'll paste the units together to get an accurate integral for an interval by pasting the elemental units together, hence using many points.

Our preliminary schemes are based on integrating interpolating polynomials.

## 2. Definitions

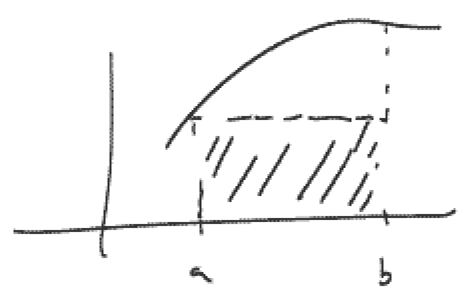
- **degree of accuracy (precision)** of a quadrature formula is the largest degree for which polynomials of that degree are integrated exactly using the formula.

- **closed Newton-Cotes formulas** use the endpoints in the approximation of the integrals, whereas **open Newton-Cotes formulas** do not use the endpoints in the approximation of the integrals.

## 3. Theorems/Formulas

We could start with interpolating constant functions (step-functions), and if we did we would begin by deriving the left- and right- rectangle





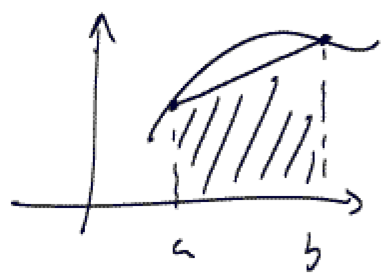
rules. But we won't! We start with the linear interpolating polynomial, from which we derive the

**Trapezoidal rule:**

$$I = \int_{x_0}^{x_1} f(x)dx = \frac{h}{2}[f(x_0) + f(x_1)] - \underline{\frac{h^3}{12}f''(\xi)}$$

error ~ $f''(\xi)$

where $\xi \in [a, b]$. It is derived by integrating the linear Lagrange intepolating polynomial:




$$I = \int_{x_0}^{x_1} f(x)dx = \int_{x_0}^{x_1} \left[\frac{(x - x_1)}{(x_0 - x_1)}f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)}f(x_1)\right] dx + \frac{1}{2}\int_{x_0}^{x_1} f''(\xi(x))(x - x_0)(x - x_1)d$$

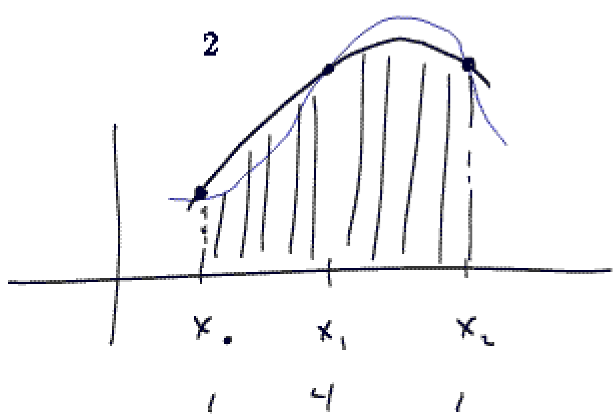
We can use the weighted mean-value theorem to write the second integral as

$$\frac{1}{2}\int_{x_0}^{x_1} f''(\xi(x))(x - x_0)(x - x_1)dx = f''(\xi)\frac{1}{2}\int_{x_0}^{x_1} (x - x_0)(x - x_1)dx = \frac{-h^3}{12}f''(\xi)$$

One interesting (but unsurprising) conclusion is that the Trapezoidal rule gets the integrals of ~~constant~~ functions right: hence its degree of accuracy (or precision) is given by 1.

**Simpson's Rule:**

can be derived similarly by integrating the Lagrange interpolating quadratic – but that means that we must use three points for the elemental unit



of Simpson's rule. The formula works out to

$$I = \int_{x_0}^{x_2} f(x)dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi)$$

This error term does not arise from the derivation using the Lagrange quadratic, however: that derivation gives an order $O(h^4)$ error term, dependent on the third derivative. It can be obtained by integrating the Taylor polynomial of degree 4 expanded about $x_1$ (not even an interpolating polynomial!), and using the centered difference formula for the first derivative. This seems curious, but – whatever works!

Astonishingly, Simpson's rule gets cubics right! Now here's one of the interesting twists: we don't have to use the quadratic geometry to understand Simpson's rule: all cubic functions interpolating three points will have the same integral between $x_0$ and $x_2$! Is that obvious, in any way?

**Example** #18, p. 196 (linear algebra wins again!)

$$\int_{x_0}^{x_2} f(x)\,dx = a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2) + k f^{(4)}(\xi)$$

I chose $x_0 = 1 \quad x_1 = 2 \quad x_2 = 3$

Linear system, three equations, three unknowns

$$\int_1^3 x\,dx = \left.\frac{x^2}{2}\right|_1^3 = a_0 1 + a_1 \cdot 2 + a_2 3$$

$$\int_1^3 x^2\,dx = \left.\frac{x^3}{3}\right|_1^3 = a_0 1 + a_1 4 + a_2 9$$

$$\int_1^3 x^3\,dx = \left.\frac{x^4}{4}\right|_1^3 = a_0 1 + a_1 8 + a_2 27$$

$$\int_1^3 x^4\,dx = \left.\frac{x^5}{5}\right|_1^3 = a_0 1 + a_1 16 + a_2 81 + \overset{f^{(4)}(x)}{24}k$$

Once those $(a_0, a_1, a_2)$ are found, use the last equation to find $k$ $(= -1/90)$