MAT360 Section Summary: 4.6

Adaptive Quadrature

1. Summary

Objective: to approximate

$$I \equiv \int_a^b f(x)dx$$

to within $\epsilon > 0$.

The trick here is the usual one, of balancing errors, and trying for a better approximation by applying several methods and then taking an appropriate combination of them.

For example, we might try a Simpson's rule on a single panel over $[a, b]$ using a step-size of $h = b - a$, and then try Simpson's composite with $h/2$:

$$I = I_h + E_h$$

where $I_h = S(a, b)$, and the error of the elemental Simpson's rule is

$$E_h = -\frac{h^5 f^{(4)}(\xi_h)}{90}$$

We now consider the error of the $h/2$ method, but used as a composite rule so that it will span $[a, b]$:

$$I = I_{h/2} + E_{h/2}$$

where $I_{h/2} = S(a, a + h) + S(a + h, b)$ and

$$E_{h/2} = -\frac{(b-a)(h/2)^4 f^{(4)}(\xi_{h/2})}{180} = -\frac{2h \cdot (h/2)^4 f^{(4)}(\xi_{h/2})}{180} = -\frac{1}{16}\frac{h^5 f^{(4)}(\xi_{h/2})}{90}$$

Now, provided $f^{(4)}$ doesn't vary wildly, we can hope that $f^{(4)}(\xi_{h/2}) \approx f^{(4)}(\xi_h)$, so that the error of the $h/2$ composite method will be about a sixteenth of the error of the elemental rule:

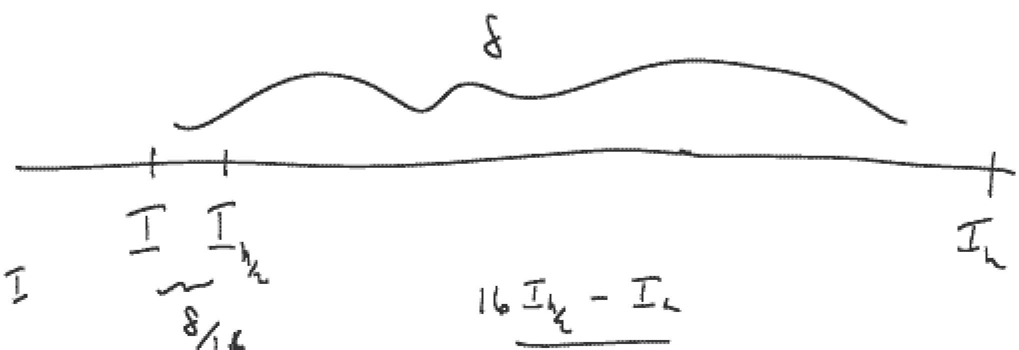
$$E_{h/2} \approx \frac{E_h}{16}$$

$16 I_{h/2} - 16 I = \delta$

$I_{h/2} - I = \frac{\delta}{16}$

$I_h - I = \delta$

$16 I_{h/2} - 16 I$
$= I_h - I$



$\frac{16 I_{h/2} - I_h}{15}$

$I = \frac{16 I_{h/2} - I_h}{15}$

Since the methods are both approximations to the same quantity, we can try to combine them to make a better approximation:

$$I \approx \frac{16 I_{h/2} - I_h}{15}$$

On the other hand, we can also to check to see if the difference is significant enough to justify using the smaller step size; if not, we can stick with the larger step size – maybe even make it larger! That is, we can **adapt** to realities "on the ground" (or on the interval, at any rate!).

So

$$I - I_{h/2} = E_{h/2} \approx \frac{1}{16} E_h = \frac{1}{16}(I - I_h)$$

If we cavalierly assume that the approximation is exact, then substituting for I we find that

$$E_{h/2} = \frac{1}{16}(I - I_h) = \frac{1}{16}(I_{h/2} + E_{h/2} - I_h)$$

or

$$16 E_{h/2} = I_{h/2} - I_h + E_{h/2}$$

Hence

$$15 E_{h/2} = I_{h/2} - I_h$$

We can easily measure the quantity on the RHS: if

$$|I_{h/2} - I_h| < 15\epsilon$$

then we conclude that

$$15|E_{h/2}| < 15\epsilon \implies |E_{h/2}| < \epsilon$$

If this condition is satisfied, then $I_{h/2}$ is sufficiently close: otherwise, we **divide and conquer**: split the error in half ($\epsilon/2$), and give one half to each of the two panels of the $h/2$ method; iterate until satisfied.