MAT360 Section Summary: 5.1

Elementary Theory of Initial-Value Problems

1. Summary

First of all we need to say a little about what differential equations and initial-value problems are, and the conditions under which they have unique solutions.

Then we need to realize that, because we're solving these problems numerically, we're not going to be solving the initial-value problem, but rather one close to that given.

In this section we get a brief view of conditions under which a "perturbed" (slightly disturbed) problem will give reasonable information about the real problem.

2. Definitions

- **differential equation**: an equation linking a function y and its derivatives and independent variables.
- **initial-value problem**: with time as the independent variable, a differential equation as well as enough initial conditions to uniquely determine the solution.
- **well-posed initial-value problem**: the initial-value problem

$$\frac{dy}{dt} = f(t,y), \quad a \leq t \leq b, \quad y(a) = \alpha \tag{1}$$

is well-posed if

(a) A unique solution y(t) exists, and

(b) For any $\epsilon > 0$ there exists a positive constant $k(\epsilon)$ such that whenever $|\epsilon_0| < \epsilon$ and $\delta(t)$ is continuous with $|\delta(t)| < \epsilon$ on $[a, b]$, a unique solution $z(t)$ to

$$\frac{dz}{dt} = f(t,y) + \delta(t), \quad a \leq t \leq b, \quad y(a) = \alpha + \epsilon_0 \tag{2}$$

exists with

$$|z(t) - y(t)| < k(\epsilon)\epsilon, \quad \text{for all } a \le t \le b.$$

The problem (2) is called a **perturbed problem** associated with the problem (1).

- **Lipschitz condition:** A function $f(t, y)$ satisfies a **Lipschitz condition** in y on a set $D \subset \mathbf{R}^2$ if a constant $L > 0$ exists with

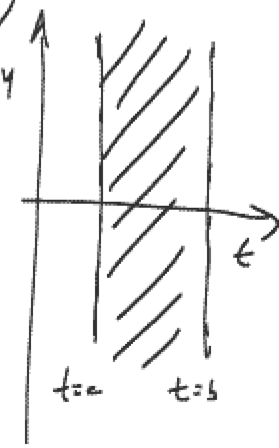
$$|f(t, y_1) - f(t, y_2)| \le L|y_1 - y_2|$$

- A set $D \subset \mathbf{R}^2$ is said to be **convex** if whenever (t_1, y_1) and (t_2, y_2) belong to D and λ is in $[0, 1]$, the point $((1-\lambda)t_1 + \lambda t_2, (1-\lambda)y_1 + \lambda y_2)$ also belongs to D.

Bad

3. Theorems/Formulas

Theorem 5.4: Suppose that $D = \{(t, y) | a \le t \le b, -\infty < y < \infty\}$ and that $f(t, y)$ is continuous on D. If f satisfies a Lipschitz condition on D in the variable y, then the initial value problem (1) has a unique solution $y(t)$ for $a \le t \le b$.



Theorem 5.3: If $f(t, y)$ is defined on a convex set $D \subset \mathbf{R}^2$. If a constant $L > 0$ exists with

$$\left|\frac{\partial f}{\partial y}(t, y)\right| \le L \quad \text{for all } (t, y) \in D,$$

then f satisfies a Lipschitz condition in y on D with Lipschitz constant L.

Theorem 5.6: Suppose that $D = \{(t, y) | a \le t \le b, -\infty < y < \infty\}$ and that $f(t, y)$ is continuous on D. If f satisfies a Lipschitz condition in y on D, then the initial value problem (1) is well-posed.

4. Properties/Tricks/Hints/Etc.

So the up-shot is that our initial-value problem satisfies a Lipschitz condition, we're in good shape: we're going to be able to get an approximation numerically.

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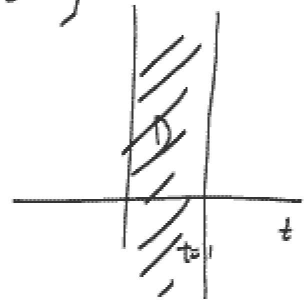
$y' = f(t, y) = y \cos t \qquad 0 \le t \le 1 \qquad y(0) = 1$

Lipschitz? $\quad \left|\frac{\partial f}{\partial y}\right| = |\cos t| \le 1 \quad \forall (t, y) \in D$

$D = \{(t,y) \mid 0 \le t \le 1 \text{ and } -\infty < y < \infty\}$



"Think method"

$$\begin{cases} y(t) = e^{\sin t} \\ y'(t) = e^{\sin t}(\sin t)' \\ \quad = e^{\sin t}\cos t = y\cos t \\ y(0) = e^0 = 1 \quad \checkmark \end{cases}$$

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separation of variables

$$\frac{dy}{dt} = y\cos t$$

$$dy = y\cos t\, dt$$

$$\frac{dy}{y} = \cos t\, dt$$

$$\int \frac{dy}{y} = \int \cos t\, dt$$

$$\ln|y| = \sin t + c$$

$$|y| = e^{\sin t + c}$$

$$y(0) = 1$$

$$|y(0)| = e^{0 + c} = e^c = 1 \quad \Rightarrow \quad c = 0$$

$$|y(t)| = e^{\sin t}$$

$$\boxed{y(t) = e^{\sin t}}$$

1a. $y' = -\frac{2}{t}y + t^2 e^t \qquad 1 \le t \le 2 \qquad y(1) = \sqrt{2}\,e$

Assume a solution of the form

$$y(t) = \sum_{i=-\infty}^{\infty} a_i t^i$$

(series solution)

$$ty' = -2y + t^3 e^t$$

Now replace everyone by a series

$$y'(t) = \left(\sum_{i=-\infty}^{\infty} a_i t^i\right)' = \sum_{i=-\infty}^{\infty} (a_i t^i)' = \sum_{i=-\infty}^{\infty} i a_i t^{i-1}$$

$$e^t = \sum_{i=0}^{\infty} \frac{t^i}{i!}$$

$$\sum_{i=-\infty}^{\infty} i a_i t^i = \sum_{i=-\infty}^{\infty} -2a_i t^i + \sum_{i=0}^{\infty} \frac{t^{i+3}}{i!}$$

The only way the equality can hold is if there's "balance" in the coefficients of each power of t.

; not to be continued!

$$\sum_{i=-\infty}^{\infty} i a_i t^i + \sum_{i=-\infty}^{\infty} 2a_i t^i - \underbrace{\sum_{i=0}^{\infty} \frac{t^{i+3}}{i!}}_{= \sum_{i=3}^{\infty} \frac{t^i}{(i-3)!}} = 0$$

$$\begin{aligned} &2a_0 \\ &+ (a_1 + 2a_1)\, t \\ &+ (2a_2 + 2a_2)\, t^2 \\ &+ \sum_{i=3}^{\infty} \left[i a_i + 2a_i - \frac{1}{(i-3)!} \right] t^i = 0 \end{aligned}$$

$$\therefore$$

$$\sum_{i=-\infty}^{-1} (2+i) a_i t^i = 0$$

$i = -2 \Rightarrow$ a_{-2} needn't be zero!

Coefficient by coefficient we must have zeros. Hence $\boxed{a_0 = 0}$

$a_1 + 2a_1 = 3a_1 = 0 \Rightarrow \boxed{a_1 = 0}$

$4a_2 = 0 \Rightarrow \boxed{a_2 = 0}$

$(2+i)a_i = \frac{1}{(i-3)!} \quad \forall i \in \{3, \dots\}$

$\therefore \boxed{a_i = \frac{1}{(2+i)(i-3)!}}$ for all other coefficients, but a_{-2}.

$$\boxed{y(t) = \sum_{i=3}^{\infty} \frac{t^i}{(2+i)(i-3)!} + \frac{a_{-2}}{t^2}}$$ is our soln.

$y(1) = \sqrt{2}\, e$

We can truncate $y(t)$ at $i \leq 15$, say, & then find a_{-2} by

$$\boxed{a_{-2} \simeq y(1) - \sum_{i=3}^{15} \frac{1}{(2+i)(i-3)!}}$$

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$y' = \frac{2}{t} y + t^2 e^t \quad 1 \leq t \leq 2 \quad y(1) = 0$

$\boxed{y(t) = t^2(e^t - e)}$ (Given in the back as soln to 1b)

$z' = \frac{2}{t} z + t^2 e^t + \delta t \quad 1 \leq t \leq 2 \quad z(1) = \varepsilon_0$

Given $\varepsilon > 0$, $\boxed{\text{find } k(\varepsilon) > 0}$ / If $|\varepsilon_0| < \varepsilon$ and $|\delta(t)| < \varepsilon$ on $[1,2]$, then

$|z(t) - y(t)| < k(\varepsilon)\varepsilon$ for $a \leq t \leq b$.

$z' - \frac{2}{t} z = t^2 e^t + \delta t$

$\frac{1}{t^2}\left[z' - \frac{2}{t} z \right] = e^t + \frac{\delta}{t}$

$$\frac{d}{dt}\left[\frac{z}{t^2}\right] = e^t + \frac{\delta}{t}$$

$$\int d\left[\frac{z}{t^2}\right] = \int \left(e^t + \frac{\delta}{t}\right) dt$$

$$\frac{z}{t^2} = e^t + \delta \ln|t| + c$$

$$z(t) = t^2\left[e^t + \delta \ln|t| + c\right]$$

$$z(1) = \varepsilon_0 = e^1 + \overset{0}{\cancel{\delta \ln 1}} + c$$

$$\boxed{c = \varepsilon_0 - e}$$

$$z(t) = t^2\left[\underline{e^t} + \delta \ln t + \varepsilon_0 - \underline{e}\right]$$

$$|z(t) - y(t)| = |t^2 \delta \ln t + t^2 \varepsilon_0|$$

$$\leq |t(\delta t)\ln t| + |t^2 \varepsilon_0|$$

We know that $|\delta t| < \varepsilon$, + that $|t| \leq 2$, and that $|\varepsilon_0| < \varepsilon$; so

$$|z(t) - y(t)| \leq |t|\,|\delta t|\,|\ln t| + |t^2|\,|\varepsilon_0|$$

$$\leq 2\,\varepsilon \ln 2 + 4\,\varepsilon$$

$$\leq \underbrace{(2\ln 2 + 4)}_{k(\varepsilon)}\,\varepsilon$$