MAT360 Section Summary: 5.2 (part II)

1. Euler's Method Error

So, using Taylor, we have that

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}y''(\xi_i)$$

Euler simply dropped the error term, to generate the succession of iterates

$$\begin{aligned} &w_0 = \alpha \\ &w_{i+1} = w_i + hf(t_i, w_i) \quad \text{for } i = 0, \dots, N \end{aligned}$$

This is a **difference equation** associated with the given differential equation. Its solution, we hope, will be relatively close to the solution of the IVP. Hope aside, how bad can things get? What's the worst that can happen? The answer is in the following theorem:

Theorem 5.9 (error bound): Suppose f is continuous and satisfies a Lipschitz condition with constant L on

$$D = \{(t, y) | a \leq t \leq b, -\infty < y < \infty\}$$

and that a constant M exists with

$$|y''(t)| \leq M, \quad \forall t \in [a, b].$$

Let $y(t)$ denote the unique solution to the IVP

$$y' = f(t, y(t)), \quad a \leq t \leq b, \quad y(a) = \alpha, \tag{1}$$

and $\{w_i\}$ be the Euler approximations. Then, for each $i = 0, \dots, N$,

$$|y(t_i) - w_i| \leq \frac{hM}{2L}\left[e^{L(ih)} - 1\right].$$

Proof: Let $y_i = y(t_i)$. Then

$$y_{i+1} - w_{i+1} = \underbrace{y(t_i)}_{= y_i} - w_i + hf(t_i, y_i) - hf(t_i, w_i) + \frac{h^2}{2}y''(\xi_i)$$

$i = 0, \ldots, N$. Therefore

$$|y_{i+1} - w_{i+1}| \le |y(t_i) - w_i| + h\overbrace{|f(t_i, y_i) - f(t_i, w_i)|}^{\le L|y_i - w_i|} + \frac{h^2}{2}M \le |y(t_i) - w_i| + hL|y_i - w_i| + \frac{h^2}{2}M$$

since f satisfies a Lipschitz condition in y. Hence

$$|y_{i+1} - w_{i+1}| \le (1 + hL)|y_i - w_i| + \frac{h^2}{2}M$$

This is related to a linear recurrence relation that will bound the error for $\epsilon_i = |y_i - w_i|$:

$$|y_{i+1} - w_{i+1}| = (1 + hL)|y_i - w_i| + \frac{h^2}{2}M$$

if you've had MAT385, then you know that we can solve these (guess and check, induction):

$$s_{i+1} = as_i + b \quad \text{with} \quad s_0 = 0$$

has solution

$$s_n = b\sum_{i=0}^{n-1} a^i = b\frac{a^n - 1}{a - 1}$$

Therefore,

$$\epsilon_n = \frac{Mh^2}{2}\frac{(1 + hL)^n - 1}{hL} \le \frac{Mh}{2L}\left[e^{nhL} - 1\right]$$

or

$$\epsilon_n = \frac{Mh^2}{2}\frac{(1 + hL)^n - 1}{hL} \le \boxed{\frac{Mh}{2L}\left[e^{(t_n - a)L} - 1\right]} \qquad (2)$$

This form shows that the error incurred by Euler's method is linear in h: for a given $T = t_n$, ϵ_n is linear in h.

2. Properties/Tricks/Hints/Etc.

One problem with the error bound (2) is that we need to have a bound on y'', and we're looking for y! This is quite a contrast to the situation when we were dealing with a known y and trying to bound higher derivatives. The chain rule may come to our rescue:

$$y''(t) = \frac{d}{dt}(y'(t)) = \frac{d}{dt}(f(t, y)) = \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t)) \cdot f(t, y(t))$$

Or it may not!;) The problem is that you will more than likely still have a term with y in it, which is unknown....

One obvious strategy for improving our Euler approximations is to make h tremendously small. This may backfire, however, due to round-off error: if we examine the perturbed difference equation

$$\begin{aligned} &u_0 = \alpha + \delta_0 \\ &u_{i+1} = u_i + hf(t_i, u_i) + \delta_{i+1} \quad \text{for } i = 0, \ldots, N \end{aligned}$$

we arrive at

Theorem 5.10: Let $y(t)$ denote the unique solution to the IVP (1), and $\{u_i\}$ the solution of the perturbed difference equation above. Then

$$|y_i - u_i| \leq \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left[e^{(t_i - a)L} - 1\right] + |\delta_0| e^{(t_i - a)L}$$

where $\delta_i \leq \delta$ for all i. The minimal error $E(h) = \frac{hM}{2} + \frac{\delta}{h}$ occurs when

$$h = \sqrt{\frac{2\delta}{M}}.$$