MAT360 Section Summary: 5.3

Higher-Order Taylor Methods

1. Summary

Rather than stop at the first term in the Taylor expansion, as Euler did,

$$y(t_{i+1}) \equiv y(t_i + h) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi_i)$$

we continue on and create an n^{th} order Taylor method:

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \ldots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$

You're perhaps wondering how we're going to compute the higher derivatives of y: well, recall the chain rule that we thought might come in handy sometimes for bounding the second derivatives in Euler's error calculations:

I hope so! :)

$$y''(t) = \frac{d}{dt}(y'(t)) = \frac{d}{dt}(f(t,y)) = \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t)) \cdot f(t, y(t))$$

or, more simply,

$$y''(t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}f$$

We can continue this for higher derivatives, although the results quickly look rather nasty: e.g.

$$y'''(t) = \frac{\partial^2 f}{\partial t^2} + 2f\frac{\partial^2 f}{\partial y \partial t} + \frac{\partial^2 f}{\partial y^2}f^2 + \frac{\partial f}{\partial y}\frac{\partial f}{\partial t} + \left(\frac{\partial f}{\partial y}\right)^2 f$$

It's actually a lot easier if you're working with a particular case and don't have to work in general. For example, if you were looking at Exercise 6b, p. 256,

$$y' = t + y$$

Then $f(t, y) = t + y$, and all higher partial derivatives of f disappear: so the general form is wasteful. We simply compute higher derivatives directly, as follows:

$$y'' = \frac{d(t+y)}{dt} = 1 + y' = 1 + t + y$$

$$y''' = \frac{d(1+t+y)}{dt} = 1 + y' = 1 + t + y = y''$$

So we've figured out quickly that all higher derivatives of y are equal. This is an interesting development: it means that y is a function of the form ae^t (which is its own derivative plus some "transiant stuff" that disappeared quickly from the higher derivatives (sounds like a polynomial to me...)). You can check that the general solution is

$$y(t) = (1+\alpha)e^t - (1+t)$$

where $y(0) = \alpha$. For 6b, p. 256, $\alpha = -1$, so $\boxed{y(t) = 1 + t}$ is the unique solution.

2. Definitions

- **Local Truncation Error**: The error made in approximating the solution of an IVP with a difference scheme of the form

  $$w_0 = \alpha$$
  $$w_{i+1} = w_i + h\phi(t_i, w_i) \quad \text{for } i = 0, \ldots, N$$

  has **local truncation error**

  $$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i)$$

  It's the error we'd make at (t_i, y_i) using the particular scheme.

For Euler's method, the local truncation error is

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - f(t_i, y_i) = \frac{h}{2}y''(\xi)$$

for some $\xi \in (t_i, t_{i+1})$.

For the Taylor method of order 2,

$$\boxed{\phi(t_i, y_i) = f(t_i, y_i) + \frac{h}{2}\left(\frac{\partial f}{\partial t}(t_i, y_i) + \frac{\partial f}{\partial y}(t_i, y_i) \cdot f(t_i, y_i)\right)}$$

so the local truncation error is

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i) = \frac{h^2}{3!} y'''(\xi)$$

for some $\xi \in (t_i, t_{i+1})$.

3. Theorems/Formulas

Theorem 5.12: If Taylor's method of order n approximates the usual IVP

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha \tag{1}$$

and if $y \in C^{n+1}[a, b]$, then the local truncation error is $O(h^n)$.