MAT360 Take-Home Final, Fall, 2005

Directions: This exam is to be taken by you, without the assistance of any other living person (you can use old dead ones like Sir Isaac Newton, however). By signing below, you indicate that your work on this exam is your own:

Name:	Your Signature:

You may use the assistance of whatever software you have used throughout the course, and you may append results to the exam. However, you are to use the space that I provided to provide your answers. This is to make my life as a grader easier; it is not just a fascistic whim.

In a similar vein, I ask that you write legibly, neatly, and well. Check your spelling. Use complete sentences. Your grader cares about good writing, and is always made a little edgy by poor spelling and writing. The first rule of students is "don't hack off the grader": please remember that!

I encourage you to work on scratch paper, and to transfer your finished work to the exam. You may refer to any appended results if you wish.

Good luck!

Problem I (20 pts total). Consider the IVP given by

$$\begin{aligned} y' &= -yt \\ y(0) &= 1 \end{aligned}$$

We're interested in the solution on the interval $0 \leq t \leq 2$.

1. The solution of this equation is $y(t) = e^{-t^2/2}$. Why do I say "The", rather than "A", solution? Demonstrate that I am correct in saying this.

2. (Problem I, continued) Use the following methods to provide an approximation to $y(h)$, where $h = .1$: In each case, bound the error that is being made, and compare the actual error to the bound.

Method	estimate	error	bound
Euler			
Taylor 2			
Taylor 3			
Taylor 4			

Problem II (20 pts total). Consider the following integral of the function $y(t) = e^{-t^2/2}$:

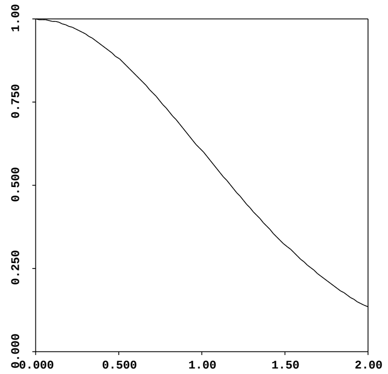
$$I = \int_0^2 e^{-t^2/2} dt$$

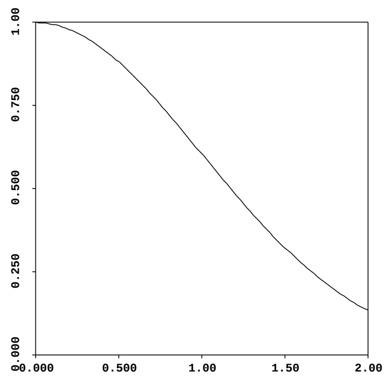
1. Approximate the value of I using the following methods (simple or composite) with $h = 1$; compute the absolute errors, and give tight error bounds for each:

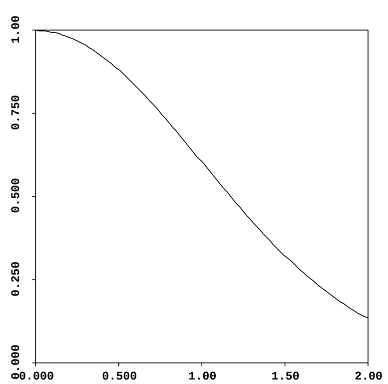
Method	value	error	bound
Trapezoidal			
Midpoint			
Simpson's			

2. Determine the largest value of h that will assure that Simpson's composite rule provides an estimate to within 1e-5 of the true value (assume true=1.1962880), and provide the corresponding estimate.

3. (Problem II continued) Demonstrate adaptive quadrature (based on Simpson's) by using the following graphs relating to the calculation of the integral I. Carry out two iterations of the process[1] with an error bound of $\epsilon = .00001$:







[1] That is, check to see if you're done right at the outset; and when you find that you're not, carry out an additional iteration.

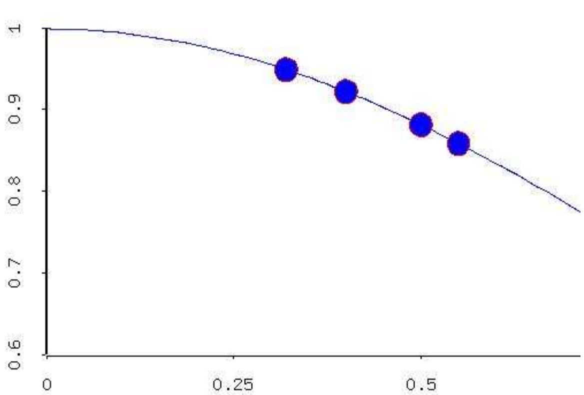
Problem III (20 pts total). Suppose that you want to use finite difference methods to approximate the derivative of $y(t) = e^{-t^2/2}$ at the point $t = 0$, with $h = .1$.

1. Compute two-point forward and backward finite-difference approximations. I've often said that, given two estimates, you have a third; use these two approximations to extrapolate a better estimate; what is the name of this estimate, and how good is it?

2. Making good use of Taylor series, find the best linear combination of the three points to

time	y value
$t_0 + .1$	$y(t_0 + .1)$
$t_0 - .08$	$y(t_0 - .08)$
$t_0 + .15$	$y(t_0 + .15)$

obtain the best approximation to $y'(t_0)$ possible (the figure below illustrates a sample case, for $t_0 = .4$):



Problem IV (20 pts total). Consider $\rho(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ and the integral

$$P(x) = \int_0^x \rho(t)dt$$

for $x \geq 0$. $P(x)$ represents the area under the graph of ρ (which is called the standard normal density function), and it varies from 0 to 1/2 as x varies from 0 to ∞. This function is very important to statisticians.

A common problem statisticians have is to find the value of x at which $P(x) = \alpha$, where $\alpha \in (0, .5)$. Suppose we take $\alpha = .475$, and suppose that we have three good guesses $x = 1.94$, $x = 1.95$, $x = 1.99$ (ranked best to worst by your boss, who is, at best, semi-reliable).

1. For each of the following methods describe how you would proceed.
   - bisection
   - secant method
   - Newton's method
   - Müller's method

2. (Problem IV continued) Now, fill in the following table, showing some work as to how you obtain

   - your first iteration using each technique,
   - and the result of using each technique until failure or the first approximation correct to within 1e-4 occurs.

Method	Initial iterate	first iterate within 1e-4	number of iterations
Bisection			
Secant			
Newton			
Müller			

Problem V (20 pts total). Because the function P defined by

$$P(x) = \int_0^x \rho(t)dt$$

is so important, yet so cumbersome to compute as defined, it is often approximated well at a variety of points, and then interpolated.

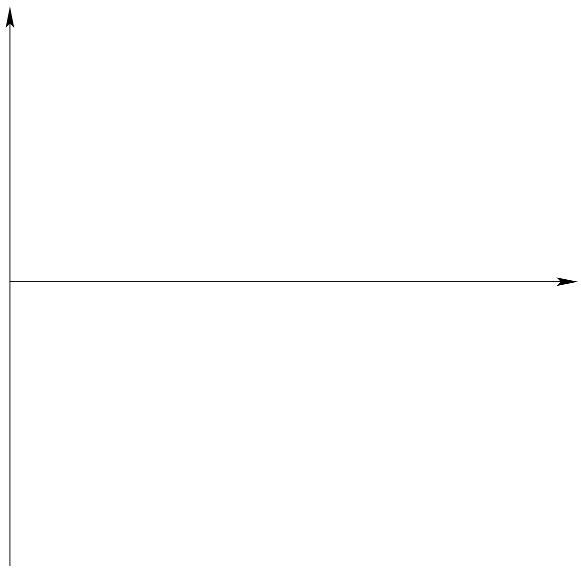
1. Fill in the following table, which we will use:

x	P (accurate to 4 decimal places)
0	
.5	
1	
1.5	
2	

2. Discuss the advantages and disadvantages of the following methods for interpolating the data:
   - Single interpolating polynomial
   - Cubic splines (natural or clamped)
   - Hermite cubic splines

3. (Problem V, continued)

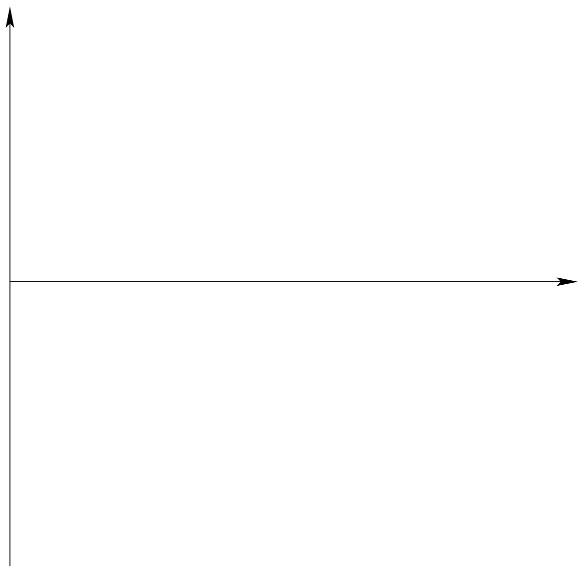
(a) Find the appropriate interpolating polynomial $P_n(x)$ to this data, and plot the difference $P(x) - P_n(x)$ on the interval [0,2].



(b) Calculate a bound on the error of the interpolating polynomial, and demonstrate that the actual error is within this bound.

4. (Problem V, continued)

(a) Find the appropriate interpolating Hermite cubic spline $H(x)$ to this data, and plot the difference $P(x) - H(x)$ on the interval [0,2].



(b) Calculate a bound on the error of the interpolating spline, and demonstrate that the actual error is within this bound.