Note Title 4/5/2005

Theorem 10.1. If $2^k - 1$ is prime $(k > 1)$ then $n = 2^{k-1}(2^k - 1)$ is perfect, & every even perfect number is of this form.

Proof:

$\Rightarrow$) Assume $2^k - 1$ is prime, & consider $n = 2^{k-1}(2^k - 1)$. One test for perfection is

$$\sigma(n) = 2n;$$

Let's see if it works out here!

$$\sigma(n) = \sigma\left[2^{k-1}(2^k-1)\right]$$

$$= \underbrace{\sigma(2^{k-1})}\,\underbrace{\sigma(2^k-1)} \quad \left(\text{since } \gcd(2^{k-1}, 2^k-1) = 1\right)$$

$$= \frac{2^{(k-1)+1}-1}{2-1} \quad \left[(2^k-1)+1\right]$$

formula
$\sigma(p^k) = \frac{p^{k+1}-1}{p-1}$

since
2^k-1 is prime, &
defn. of σ

$$= (2^k - 1)\, 2^k = 2\left[2^{k-1}(2^k - 1)\right] = 2n$$

$\Leftarrow$: Assume that n is a perfect, even number. We want to show that $2^k - 1$ is prime.

We can write

$$n = 2^{k-1} \cdot m \qquad k \geq 2$$

where m is odd.

We know that $\sigma(n) = 2n$, since n is perfect. So

$$\sigma(n) = \sigma(2^{k-1})\,\sigma(m) \quad \left(\text{since } \gcd(2^{k-1}, m) = 1\right)$$

$$= \underbrace{(2^k - 1)}_{\text{from the formula}} \cdot \sigma(m) \quad \underbrace{= 2n}_{\text{since } n \text{ is perfect}} = 2 \cdot 2^{k-1} \cdot m$$

$$= 2^k \cdot m$$

from which we can conclude that

$$2^k - 1 \mid m \quad \left(\text{since } \gcd(2^k-1, 2^k) = 1\right)$$

So we can write

$$m = \underline{(2^k - 1) M}$$

$$(2^k - 1)\sigma(m) = 2^k m$$ from above

$$\cancel{(2^k - 1)}\sigma(m) = 2^k \cancel{(2^k - 1)} M$$

$$\sigma(m) = 2^k M$$

$$\geq m + M = (2^k - 1)M + M$$

$$= 2^k M$$

So, in fact, $\sigma(m) = m + M$

Therefore m is prime, & $M = 1$.

So $m = 2^k - 1$, and

$$n = 2^{k-1}(2^k - 1).$$

Q.E.D.

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(a) Consider $n = p^k$. Check for perfection:

$$\sigma(n) = 2n \ ?$$

~~By~~ By contradiction: assume that n's perfect.

$$\sigma(p^k) = 2p^k = \frac{p^{k+1}-1}{p-1}$$

So, clearing the denominator,

$$2(p^{k+1} - p^k) = p^{k+1} - 1$$

or

$$p^{k+1} - 2p^k + 1 = 0$$

$$p^k \underbrace{(p-2)}_{\geq 0} + 1 = 0$$

so there can be no prime solutions.

Contradiction. p^k isn't perfect!

(b) No perfect square can be a perfect number.

Let

$$n = p_1^{2k_1} \cdots p_r^{2k_r}$$

& assume n is perfect: $\sigma(n) = 2n$.

$$\sigma(n) = \prod \underbrace{\frac{p_i^{2k_i+1} - 1}{p_i - 1}}_{\sigma(p_i^{2k_i})} = 2 \cdot n$$

Hence $2 \mid \sigma(p_i^{2k_i})$ for some i.