# Number Theory Section Summary: 4.3
Special Divisibility Tests

## 1. Summary

You all knew that an even integer is divisible by two, and you probably all knew the rule for divisibility by three: that if the sum of the digits is divisible by three, then so is the number itself. Do you know the rule for divisibility by 9, or by 11?

These are curious rules, because they're based in part on the base of the number system in use. As we saw in the case of the sieve of Eratosthenes, the spiraling patterns that developed were a consequence of the wrapping of the integers at every 10th.

This section delves into bases, and proves that these rules for divisibility work.

## 2. Definitions

*base $b$ place-value notation*: the representation of a number as as string of coefficients of powers of $b$ (e.g. decimal notation, or binary - base 2 - number system):

$$N = (a_m a_{m-1} \cdots a_2 a_1 a_0)_b$$

which stands for

$$N = a_m b^m + a_{m-1} b^{m-1} + \cdots + a_2 b^2 + a_1 b^1 + a_0 b^0$$

where $a_i \in \{0, 1, \ldots, b-1\}$ and $a_m \neq 0$.

## 3. Theorems

**Theorem:** Given any integer $b > 1$, any integer may be written uniquely in base $b$ place-value notation.



$N \in \mathbb{Z}$

$N = q_1 b + r_1 \qquad 0 \leq r_1 < b$

If $q_1 \geq b$, then

$$q_1 = q_2 b + r_2$$

$$N = (q_2 b + r_2) b + r_1 = q_2 b^2 + r_2 b + r_1$$

repeat as necessary ....

Proof: repeated applications of the division algorithm.

**Theorem 4.4**: Let $P(x) = \sum_{k=0}^{m} c_k x^k$ be a polynomial function of $x$ with integral coefficients $c_k$. If $a \equiv b(\text{mod n})$, then $P(a) \equiv P(b)(\text{mod n})$.

**Corollary**: If $a$ is a solution of the congruence $P(x) \equiv 0(\text{mod n})$, and $a \equiv b(\text{mod n})$, then $b$ is also a solution.

If $P(a) \equiv 0 \pmod n$
then $P(b) \equiv 0 \pmod n$

**Theorem 4.5/4.6**: Let

$$N = a_m 10^m + a_{m-1} 10^{m-1} + \cdots + a_2 10^2 + a_1 10 + a_0$$

be the decimal expansion of positive integer $N$, $0 \leq a_k < 10$, and let $S = a_0 + a_1 + \ldots + a_m$. Then

- $9|N \iff 9|S$.
- Let $T = a_0 - a_1 + a_2 - \ldots + (-1)^m a_m$. Then $11|N \iff 11|T$.

## 4. Properties/Tricks/Hints/Etc.

Often "the trick" to solving the problems involves

- figuring out which $n$ in "modulo $n$" we need, or
- figuring out how to rewrite things modulo $n$ so that good things happen.

#6a, p. 73

$N \in \mathbb{Z}$ is divisible by 2 $\iff$

last decimal digit is 0, 2, 4, 6, 8

$$N = \underbrace{a_m 10^m + a_{m-1} 10^{m-1} + \ldots + a_2 10^2 + a_1 10}_{} + a_0$$



$$10 \equiv 0 \pmod 2$$

$$N \equiv a_0 \pmod 2$$

$$N \equiv 0 \pmod 2 \iff a_0 \equiv 0 \pmod 2$$

#6b $10 \equiv 1 \pmod 3$

$$N \equiv a_m + a_{m-1} + \dots + a_1 + a_0 \pmod 3$$

$$N \equiv 0 \pmod 3 \iff$$

$$\sum_{i=0}^{m} a_i \equiv 0 \pmod 3$$

18a) $N$, $M$ (digits of $N$ in reverse order).

$$N = (a_m \cdots a_0)_{10} = a_m 10^m + \dots + a_0$$

$$M = (a_0 \cdots a_m)_{10} = a_0 10^m + \dots + a_m$$

$$N - M = (a_m - a_0)10^m + (a_{m-1} - a_1)10^{m-1} + \dots +$$

$$(a_1 - a_{m-1})10 + (a_0 - a_m)$$

Sum the digits:

$$= 0 \equiv 0 \pmod 9$$

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Generalize the result (Isto's Theorem):

Given $N \in \mathbb{N}$ & $M$ a number obtained by permuting the digits of $N$. Then

$N - M$ is divisible by 9.

$$N - M = (a_m - b_m)10^m + (a_{m-1} - b_{m-1})10^{m-1} + \dots$$

$$+ (a_1 - b_1)10 + (a_0 - b_0)$$

where $b_i = a_j$ for some $j \in \{0, \dots, m\}$, a permutation of the digits of $N$.

$$\equiv (a_m - b_m) + (a_{m-1} - b_{m-1}) + \dots + (a_1 - b_1) + (a_0 - b_0) \pmod 9$$

$$\equiv \sum_{k=0}^{m} a_k - \sum_{k=0}^{m} b_k \pmod n$$

$$\equiv 0 \pmod n$$

So $9 \mid N - M$.