Number Theory Section Summary: 6.1
Number-Theoretic Functions

1. Summary

We encounter two interesting number-theoretic functions, τ and σ, and discover an interesting relationship between these and the prime factorization of a number.

The concept of a multiplicative function is also introduced, which will prove useful (now and later on).

2. Definitions

Number-theoretic function: any function whose domain is the set of positive integers

Definition 6.1: Given a positive integer n, let $\tau(n)$ denote the number of positive divisors of n, and $\sigma(n)$ denote the sum of those divisors.

The notation

$$\sum_{d|n} f(d)$$

means "sum the values of f as d runs over the divisors of n". Given that, then

$$\tau(n) = \sum_{d|n} 1$$

and

$$\sigma(n) = \sum_{d|n} d$$

Problem: Evaluate $\tau(24)$ and $\sigma(24)$.

Problem: Evaluate $\tau(240)$ and $\sigma(240)$.

Problem: What are $\tau(p)$ and $\sigma(p)$ when p is prime?

Problem: #15, p. 110

Problem: How do $\tau(4)\tau(6)$ and $\tau(24)$ compare?

Definition 6.2: A number-theoretic function is said to be **multiplicative** if

$$f(mn) = f(m)f(n)$$

whenever $\gcd(m, n) = 1$

Examples: $f(n) = 1$, $f(n) = n$.

By induction,

$$f(n_1 n_2 \cdots n_r) = f(n_1)f(n_2)\cdots f(n_r)$$

whenever the n_i are pairwise relatively prime. Hence, a multiplicative function is completely determined for n once its values on the prime powers of the factorization of n are known:

$$f(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}) = f(p_1^{k_1})f(p_2^{k_2})\cdots f(p_r^{k_r})$$

Example: #17, p. 110

3. Theorems

Theorem 6.1 If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of $n > 1$, then the positive divisors of n are precisely those integers of the form $d = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where $0 \le a_i \le k_i$ for i in $\{1, \ldots, r\}$.

Theorem 6.2 If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of $n > 1$, then

(a)

$$\tau(n) = (k_1 + 1)(k_2 + 1)\cdots(k_r + 1)$$

and

$$= (1+p_1+\dots+p_1^{k_1})(1+p_2+\dots+p_2^{k_2})\dots (1+p_r+p_r^{k_r})$$

(b)

$$\sigma(n) = \frac{p_1^{k_1+1}-1}{p_1-1}\frac{p_2^{k_2+1}-1}{p_2-1}\cdots\frac{p_r^{k_r+1}-1}{p_r-1}$$

The proof of the first is a counting argument, and the second uses a sum of a geometric series and a neat decomposition.

The notation

$$\prod_{i=1}^{r} f(i)$$

means "multiply the values of f as i runs over from 1 to r". Given that, then

$$\tau(n) = \prod_{i=1}^{r}(k_i+1)$$

and

$$\sigma(n) = \prod_{i=1}^{r}\frac{p_i^{k_i+1}-1}{p_i-1}$$

Let's check for $n = 240$.

Theorem 6.3 The functions τ and σ are multiplicative functions.

Lemma If $\gcd(m,n)=1$, then the set of positive divisors of mn consists of all products $d_1 d_2$, where $d_1|m$, $d_2|n$, and $\gcd(d_1,d_2)=1$; furthermore these products are all distinct.

Theorem 6.4 If f is a multiplicative function and F is defined by

$$F(n) = \sum_{d|n} f(d)$$

then F is also multiplicative.

Corollary: the functions τ and σ are multiplicative functions.

$$\tau(m\cdot n) = \tau(\overbrace{p_1^{k_1}\cdots p_r^{k_r}\cdot q_1^{l_1}\cdots q_s^{l_s}}^{r+s \text{ distinct primes}}) \quad \text{since } \gcd(m,n)=1$$

$$= \underbrace{(k_1+1)(k_2+1)\cdots(k_r+1)}_{=\tau(m)}\underbrace{(l_1+1)(l_2+1)\cdots(l_s+1)}_{\tau(n)}$$