Number Theory Section Summary: 6.1
Number-Theoretic Functions

1. Summary

We encounter two interesting number-theoretic functions, τ and σ, and discover an interesting relationship between these and the prime factorization of a number.

The concept of a multiplicative function is also introduced, which will prove useful (now and later on).

2. Definitions

Number-theoretic function: any function whose domain is the set of positive integers

Definition 6.1: Given a positive integer n, let $\tau(n)$ denote the number of positive divisors of n, and $\sigma(n)$ denote the sum of those divisors.

The notation

$$\sum_{d|n} f(d)$$

means "sum the values of f as d runs over the divisors of n". Given that, then

$$\tau(n) = \sum_{d|n} 1$$

and

$$\sigma(n) = \sum_{d|n} d$$

index set of divisors

divisors of 24 are: $\{1, 2, 3, 4, 6, 8, 12, 24\}$

Problem: Evaluate $\tau(24)$ and $\sigma(24)$.

$\tau(24) = 8 \qquad \sigma(24) = 60$

Problem: Evaluate $\tau(240)$ and $\sigma(240)$.

$\tau(240) = 20 \qquad \sigma(240) = 744$

Problem: What are $\tau(p)$ and $\sigma(p)$ when p is prime?

$\{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 40, 48, 60, 80, 120, 240\}$

$\tau(p) = 2$

$\sigma(p) = p + 1$

#15 $n, n+2$ twin primes

$$\sigma(n+2) = n+3 = (n+1)+2$$
$$= \sigma(n) + 2$$

Problem: #15, p. 110

Problem: How do $\overset{3\cdot 4}{\tau(4)\tau(6)}$ and $\overset{8}{\tau(24)}$ compare?

Definition 6.2: A number-theoretic function is said to be **multiplicative** if

$$f(mn) = f(m)f(n)$$

whenever $\gcd(m, n) = 1$

Examples: $f(n) = 1$, $f(n) = n$. Let $f(n) = n$; then $f(m \cdot n) = m \cdot n = f(m) \cdot f(n)$ ✓

By induction,

$$f(n_1 n_2 \cdots n_r) = f(n_1)f(n_2)\cdots f(n_r)$$

whenever the n_i are pairwise relatively prime. Hence, a multiplicative function is completely determined for n once its values on the prime powers of the factorization of n are known:

$$f(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}) = f(p_1^{k_1})f(p_2^{k_2})\cdots f(p_r^{k_r})$$

Example: #17, p. 110 Show that $f(n) = n^k$ is multiplicative, $k \in \mathbb{Z}$. $f(m \cdot n) = (m \cdot n)^k = m^k \cdot n^k$ $= f(m) \cdot f(n)$ ✓

3. Theorems

Theorem 6.1 If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of $n > 1$, then the positive divisors of n are precisely those integers of the form $d = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where $0 \le a_i \le k_i$ for i in $\{1, \ldots, r\}$.

Theorem 6.2 If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of $n > 1$, then

(a) $\tau(240)$? $240 = 2^4 \cdot 3 \cdot 5$

$$\tau(n) = (k_1+1)(k_2+1)\cdots(k_r+1)$$

$\tau(240) = (4+1)(1+1)(1+1) = 20$

and

$$\sigma(n) = \underbrace{(1 + p_1^1 + \ldots + p_1^{k_1})}_{\frac{p_1^{k_1+1}-1}{p_1-1}}(1 + p_2^1 + \ldots + p_2^{k_2}) \cdots (1 + p_r^1 + \ldots + p_r^{k_r})$$

(every divisor appears exactly once)

(sum of a geometric series)

(b)

$$\sigma(n) = \frac{p_1^{k_1+1}-1}{p_1-1}\frac{p_2^{k_2+1}-1}{p_2-1}\cdots\frac{p_r^{k_r+1}-1}{p_r-1}$$

$\sigma(240)$?

$$= \frac{2^5-1}{2-1}\,\frac{3^2-1}{3-1}\,\frac{5^2-1}{5-1}$$

The proof of the first is a counting argument, and the second uses a sum of a geometric series and a neat decomposition.

$= 31 \cdot 4 \cdot 6$

$= 744$

The notation

$$\prod_{i=1}^{r} f(i)$$

means "multiply the values of f as i runs over from 1 to r". Given that, then

$$\tau(n) = \prod_{i=1}^{r}(k_i + 1)$$

and

$$\sigma(n) = \prod_{i=1}^{r}\frac{p_i^{k_i+1}-1}{p_i-1}$$

Let's check for $n = 240$.

Theorem 6.3 The functions τ and σ are multiplicative functions.

Lemma If $\gcd(m,n) = 1$, then the set of positive divisors of mn consists of all products d_1d_2, where $d_1|m$, $d_2|n$, and $\gcd(d_1,d_2) = 1$; furthermore these products are all distinct.

Theorem 6.4 If f is a multiplicative function and F is defined by

$$F(n) = \sum_{d|n} f(d)$$

then F is also multiplicative.

Corollary: the functions τ and σ are multiplicative functions.