

Highlights, Section 1.2: Propositional Logic

- **Propositional wff:** represent some sort of argument, to be tested, or proven, by **propositional logic**.
- **valid arguments**, e.g.

$$P_1 \text{ and } P_2 \text{ and } \dots \text{ and } P_n \rightarrow Q$$

have **hypotheses** (we suppose that the P_i are true), and a **conclusion** (Q). To be *valid*, this argument must be a tautology (always true). To be an *argument*, Q must not be identically true (i.e. a fact, in which case the hypotheses would be irrelevant!).

- **Proof Sequence:** a sequence of wffs in which every wff is a hypothesis or the result of applying the formal system's derivation rules (truth-preserving rules) in sequence.

Objective: to reach the conclusion Q from the hypotheses P_1, P_2, \dots, P_n .

- **Types of derivation rules:**
 - **Equivalence rules** (see Table 1.12, p. 23):

we can substitute equivalent wffs in a proof sequence. One way of showing that two wffs are equivalent is via their truth tables.

- commutative
- associative
- De Morgan's laws
- implication
- double negation

Implication seems somewhat unusual, but it is suggested by [Exercise 6a, section 1.1](#). You're asked to prove it in Practice 9, p. 23. That is, prove that

$$(P \rightarrow Q) \leftrightarrow (P' \text{ or } Q)$$

is a tautology. How would you do it?

- **Inference rules:** from given hypotheses, we can deduce certain conclusions (see Table 1.13, p. 24)
 - **modus ponens:**
If Q follows from P , and P is true, then so is Q .
 - **modus tollens:**
If Q follows from P , and Q is false, then so is P .
 - **conjunction:**
If Q is true, and P is true, then they're both true together.
 - **simplification:**
If both Q and P are true, then they're each true separately.
 - **addition:**
If P is true, then either P or Q is true.

Practice 10, p. 24. Also give step 4!

1. $(A \wedge B)' \rightarrow C$ hyp
 2. C' hyp
 3. $(A \wedge B)''$ 1, 2 mt
 4. $A' \vee B'$ 3 de Morgan
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#27 $P \wedge P' \rightarrow Q$

1. P hyp
2. P' hyp
3. $P \vee Q$ 1, add
4. $P' \vee Q$ 2, add
5. $P \rightarrow Q$ 4, imp
6. Q 1, 5 mp

For a more elaborate example, let's look at [#27, p. 32](#), which shows that one can prove anything if one introduces a contradiction (e.g. [the mensa quiz](#)). Also called an **inconsistency**.

- The difference between equivalence rules and inference rules is that equivalence rules are bi-directional (work both ways), whereas some inference rules are uni-directional (work in only one direction - this is what inference is all about: from this we can infer that, but we cannot necessarily infer this from that!).

Notice that in the table 1.14 (p. 31) some rules appear twice: two uni-directionals can make a bi-directional!

Note for your homework: you are not allowed to invoke the rule that you are trying to prove! Notice that the entries in this table are followed by exercise numbers: it is in those exercises that the results are obtained!

- **Deduction method:** if we seek to prove an implication, we can simply add the hypothesis of this conclusion implication to the hypothesis of the argument, and prove the conclusion of the remaining implication:

$$P_1 \text{ and } P_2 \text{ and } \dots \text{ and } P_n \rightarrow (R \rightarrow S)$$

can be replaced by

$$P_1 \text{ and } P_2 \text{ and } \dots \text{ and } P_n \text{ and } R \rightarrow S$$

If you're interested in seeing why this rule works, you might try exercise 45, p. 33, but think of it this way: we're interested in assuming that all the P_i are true, and see if we can deduce the implication $R \rightarrow S$. If R is false, then the implication is true. The only potentially problematic case is where R is true, and S is false. Then what we want to know is: given that

$$P_1 \text{ and } P_2 \text{ and } \dots \text{ and } P_n \text{ and } R$$

are true, is S true?

Exercise #32, p. 32

- **Hypothetical syllogism:**

$$\text{if } P \rightarrow Q \text{ and } Q \rightarrow R, \text{ then } P \rightarrow R.$$

(and see a whole long list of rules in Table 1.14). This rule might be referred to as **transitivity**.

A new rule is created each time we prove an argument; but we don't want to create so many rules that we keel over under their weight! Keep just a few rules in view, and learn how to use them well....

- Our goal may well be to turn a "real argument" into a symbolic one. This allows us to test whether the argument is sound (that is, that the conclusion follows from the hypotheses).

Exercise #39, p. 32.

- The propositional logic system is complete and correct:
 - **complete:** every valid argument is provable.
 - **correct:** only a valid argument is provable.

The derivation rules are truth-preserving, so correctness is pretty clear; completeness is not! How can we tell if we can prove every valid argument?!

$$\#32 \quad (A' \rightarrow B') \wedge (A \rightarrow C) \rightarrow \underline{(B \rightarrow C)}$$

1. $A' \rightarrow B'$ hyp
2. $A \rightarrow C$ hyp
3. B hyp (deduction method)
4. $B \rightarrow A$ 1, cont
5. $B \rightarrow C$ 4, 2 hs
6. C 3, 5 mp

Q.E.D

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- J - Jane is more popular
E - Jane will be elected
C - Craig will resign

$$(J \rightarrow E) \wedge (J \rightarrow C) \longrightarrow \\ (J \rightarrow (E \wedge C))$$

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