Number Theory Section Summary: 13.1
Fibonacci Numbers

1. Summary

Leonardo de Pisa (1180-1250?), better known as Fibonacci, wrote the *Liber Abaci*, in which he included a problem about rabbits:

A man put one pair of rabbits in a certain place entirely surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year, if the nature of these rabbits is such that every month each pair bears a new pair which from the second month on becomes productive?

Ignoring the terrible incestuous implications, the resulting sequence of numbers of pairs of rabbits is known as the **Fibonacci numbers**:

$$\ldots, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots$$

This works out to the recursive sequence

$$\boxed{u_n = u_{n-1} + u_{n-2}}$$

for $n \geq 3$, where $u_1 = u_2 = 1$, the first known recursive definition in mathematics.

2. **Theorems**

An important result which we will need in the following theorems is this:

$$u_{m+n} = u_{m-1}u_n + u_m u_{n+1} \tag{1}$$

Proof: by induction on n. Given $m \geq 2$. Consider

Base: $n=1$; demonstrate that

$$u_{m+1} = u_{m-1}\underbrace{u_1}_{1} + u_m \underbrace{u_{1+1}}_{1}$$

$$u_{m+1} = u_{m-1} + u_m \text{ by defn.} \checkmark$$

$\Rightarrow$: Assume true for n up to k. Consider

$$u_{m+k+1} = u_{m+k} + u_{m+k-1}$$

$$= u_{m-1}u_k + u_m u_{k+1}$$

$$+ u_{m-1}u_{k-1} + u_m u_{k-1+1}$$

$$= u_{m-1}u_{k+1} + u_m u_{k+2}$$

$$= u_{m-1} u_{k+1} + u_m u_{(k+1)+1}$$

$\therefore$ the result holds by induction.

Theorem 13.1: For the Fibonacci sequence, $\gcd(u_n, u_{n+1}) = 1$ for every $n \geq 1$.

Proof: direct, and using lemma, p. 27.

Direct:
$$u_{n+1} = u_n + u_{n-1}$$
$$u_{n-1} = u_{n+1} - u_n$$

So $\gcd(u_n, u_{n+1}) \mid u_{n-1}$

Hence
$$\begin{aligned} \gcd(u_{n+1}, u_n) &\leq \gcd(u_n, u_{n-1}) \\ &\leq \gcd(u_{n-1}, u_{n-2}) \\ &\vdots \\ &\leq \gcd(u_2, u_1) = 1 \end{aligned}$$

$\therefore \gcd(u_{n+1}, u_n) = 1$

$u_{12} = 144$

$u_6 = 8$

$u_4 = 3$

$u_3 = 2$

$u_2 = 1$

$u_1 = 1$

Theorem 13.2: For $m \geq 1$ and $n \geq 1$, $u_m | u_{mn}$.

Proof: by induction on n (straightforward, using (??)).

$=$ Eq. 1

Base: $n = 1$

$$u_m \mid u_m \checkmark$$

$\Rightarrow$: Assume true for $n \leq k$.

Consider

$$u_{m(k+1)} = u_{mk+m}$$

$$= u_{mk-1} u_m + u_{mk} \cdot u_{m+1}$$

$u_m \mid u_m$ (of course) + $u_m \mid u_{mk}$ by hypothesis;

hence $u_m \mid u_{m(k+1)}$ $\checkmark$

Q.E.D.

Lemma: If $m = qn + r$, then $\gcd(u_m, u_n) = \gcd(u_r, u_n)$

Let $d = \gcd(u_m, u_n)$

$$u_m = u_{qn+r}$$

$$= u_{qn-1} u_r + u_{qn} \cdot u_{r+1}$$

Rewrite as

$$u_m - u_{qn} \cdot u_{r+1} = u_{qn-1} \cdot u_r$$

$d \mid u_m$ & $d \mid u_{qn}$

(because $u_n \mid u_{qn}$)

so $d \mid u_{qn-1} \cdot u_r$

$\therefore$ $d \mid u_r$ ($d \nmid u_{qn-1}$ because u_{qn} & u_{qn-1} are relatively prime)

$\therefore$ $d \mid \gcd(u_r, u_n)$

Theorem 13.3: The greatest common divisor of two Fibonacci numbers is again a Fibonacci number; specifically $\gcd(u_m, u_n) = u_d$ where $d = \gcd(m, n)$.

Euclidean Algorithm:

$$m = q_1 n + r_1$$

$$n = q_2 r_1 + r_2$$

$$r_1 = q_3 r_2 + r_3$$

$$\vdots$$

$$r_{k-1}$$

Corollary: In the Fibonacci sequence, $u_m | u_n$ if and only if $m|n$ for $n \geq m \geq 3$.

3. Properties/Tricks/Hints/Etc.

- For every prime p, there are infinitely many Fibonacci numbers that are divisible by p, equally spaced along the Fibonacci sequence.
- It is not known if there are infinitely many prime Fibonacci numbers.