Section 1.4: Predicate Logic

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Abstract

We now consider the logic associated with predicate wffs, including a new set of derivation rules for demonstrating validity (the analogue of tautology in the propositional calculus).

1 Derivation rules

- First of all, all the rules of propositional logic still hold. Whew! Propositional wffs are simply boring, variable-less predicate wffs.

- Our author suggests the following "general plan of attack":

  - strip off the quantifiers
  - work with the separate wffs
  - insert quantifiers as necessary

  Now, how may we legitimately do so?

- New rules for predicate logic: in the following, you should understand by the symbol x in $P(x)$ an expression with free variable x, possibly containing other (quantified) variables: e.g.

$$P(x) = (\forall y)(\exists z)Q(x, y, z) \tag{1}$$

– **Universal Instantiation**: from $(\forall x)P(x)$ deduce $P(t)$.

Caveat: t must not already appear as a variable in the expression for $P(x)$: in the equation above, (1), it would not do to use $P(y)$ or $P(z)$, as they appear in the expression already.

Example: Practice 22, p. 48

$$(\forall x)[P(x) \to R(x)] \wedge [R(y)]' \to [P(y)]'$$

1. $(\forall x)[P(x) \to R(x)]$ hyp
2. $[R(y)]'$ hyp
3. $P(y) \to R(y)$ 1, ui
4. $[P(y)]'$ 3, 2 mt

– **Existential Instantiation**: from $(\exists x)P(x)$ deduce $P(t)$.

Caveat: t must be introduced for the first time (so do these early in proofs). You can do a universal instantiation which also uses t after an existential instantiation with t, but not *vice versa* (e.g. Example 27).

Example: Ex. #11, p. 58 (start).

$$(\forall x)P(x) \wedge (\exists x)Q(x) \to (\exists x)[P(x) \wedge Q(x)]$$

1. $(\forall x)P(x)$ hyp
2. $(\exists x)Q(x)$ hyp
3. $Q(x)$ 2, ei
4. $P(x)$ 1, ui
5. $P(x) \wedge Q(x)$ 3, 4, con
6. $(\exists x)[P(x) \wedge Q(x)]$ 5, eg

– **Universal Generalization**: from $P(x)$ deduce $(\forall x)P(x)$.

Caveats:

* $P(x)$ hasn't been deduced by existential instantiation from any hypothesis in which x was free, and
* $P(x)$ hasn't been deduced by existential instantiation from another wff in which x was free.

Example: Ex. #17, p. 58

$$(\forall x)[P(x) \to Q(x)] \to [(\forall x)P(x) \to (\forall x)Q(x)]$$

1.	$(\forall x)[P(x) \to Q(x)]$	hyp
2.	$P(x) \to Q(x)$	1, ui
3.	$(\forall x)P(x)$	deduction method
4.	$P(x)$	3, ui
5	$Q(x)$	2, 4 mp
6.	$(\forall x)\,Q(x)$	5, ug
7.	$(\forall x)P(x) \to (\forall x)\,Q(x)$	deduction method.

(Note: the deduction method still applies, of course.)

– **Existential Generalization**: from $P(a)$ deduce $(\exists x)P(x)$.

Caveat: x must not appear in $P(a)$.

Example: Ex. #11, p. 58 (finish).

Look at the three proofs using a temporary hypothesis (Examples #31, and 32(a,b)). Notice how the introduction of the temporary hypothesis ends with an implication, which is then useful for the continuation of the proof.

Example: Practice 25, p. 52

$(\forall x)[B(x) \vee C(x) \to A(x)] \to (\forall x)[B(x) \to A(x)]$

1. $(\forall x)[B(x) \vee C(x) \to A(x)]$ hyp
2. $B(x) \vee C(x) \to A(x)$ 1, ui
3. $B(x)$ temp hyp
4. $B(x) \vee C(x)$ 3, add
5. $A(x)$ 2, 4, mp
6. $B(x) \to A(x)$ temp hyp discharged
7. $(\forall x)[B(x) \to A(x)]$ ug.

So now, how would we demonstrate that

$$(\exists y)[P(x) \to Q(x,y)] \iff [P(x) \to (\exists y)Q(x,y)]$$

Show in both directions

$(\exists y)[P(x) \to Q(x,y)] \to [P(x) \to (\exists y)Q(x,y)]$

1. $(\exists y)[P(x) \to Q(x,y)]$ hyp
2. $P(x) \to Q(x,y)$ 1, ei
3. $P(x)$ deduction method
4. $Q(x,y)$ 2, 3 mp
5. $(\exists y)Q(x,y)$ 4, eg. Q.E.D.

$[P(x) \to (\exists y)Q(x,y)] \to (\exists y)[P(x) \to Q(x,y)]$

1. $P(x) \to (\exists y)Q(x,y)$ hyp
2. $P(x)$ temp hyp
3. $(\exists y)Q(x,y)$ 1, 2, mp

4. $Q(x,y)$ 3, ei

5. $P(x) \to Q(x,y)$ temp hyp discharged

6. $(\exists y)[P(x) \to Q(x,y)]$ 5, e.g. Q.E.D.

$$[P(x) \to (\exists y)Q(x,y)] \wedge \underline{[P(x) \to Q(x,y)]} \to (\exists y)[P(x) \to Q(x,y)]$$

new ammo!

$\therefore$ The two wffs are equivalent.