Section 2.4: Recursion and Recurrence Relations

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Abstract

In this section we examine the definition and multiple applications of recursion, and encounter many examples. We also solve one type of linear recurrence relation to give a general closed-form solution.

1 Recursion

A recursive definition is one in which

1. A basis case (or cases) is given, and

2. an inductive or recursive step describes how to generate additional cases from known ones.

Example: the Factorial function sequence: $n!$

1. $F(0) = 1$, and $\boxed{0! = 1}$

2. $F(n) = nF(n-1)$.

$1! = 1 \cdot 0! = 1 \cdot 1 = 1$

$F(1) = 1 \cdot F(0)$
$= 1 \cdot 1 = 1$

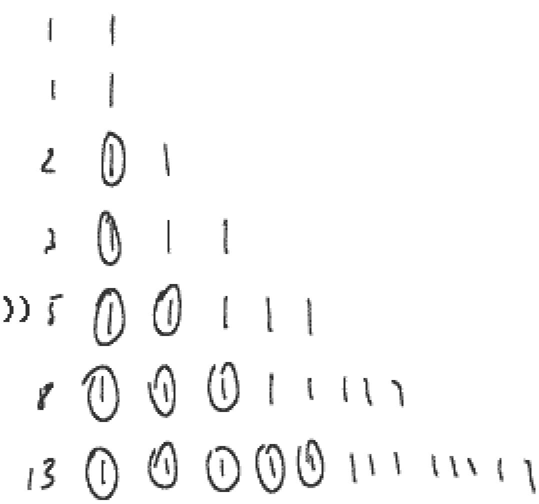
$2! = 2 \cdot 1! = 2 \cdot 1 = 2$

$3! = 3 \cdot 2! = 3 \cdot 2 = 6$

Note: This method of defining the Factorial function obviates the need to "explain" the fact that $F(0) = 0! = 1$. For that reason, it's better than defining the Factorial function as "the product of the first n positive integers," which it is from $n = 1$ on....

In this section we encounter examples of several different objects which are defined recursively (See Table 2.5, p. 131):

- **sequences** – an enumerated list of objects (e.g. Fibonacci numbers - Practice 12, p. 122 - history, #32, p. 142)



I'm very fond of lisp:

$F(0) = 1$
$F(1) = 1$
$F(n) = F(n-1) + F(n-2)$

```
(defun fib(n)
  (case n
        (0 1)
        (1 1)
        (t (+ (fib (- n 1)) (fib (- n 2))))
        )
  )
> (fib 4)
5
> (mapcar #'fib (iseq 0 8))
(1 1 2 3 5 8 13 21 34)
```

Note: The differences in examples #31 and #32 illustrate why you want to stop and think before you attempt a proof!

$F(n+4) = 3F(n+2) - F(n)$

- **sets** (e.g. finite length and palindromic strings - Example 34 and Practice 16 and 17, pp. 124-125)

Alphabet A

$A^* = \{ s \mid s \text{ is a finite string from } A \}$.

Base case: $\begin{cases} \lambda \text{ the empty string is in } A^* \\ \text{any single character from } A \text{ is in } A^* \end{cases}$

If x + y are strings in A^*, so is xy.

Practice 16: $x = 1011$ and $y = 001$

xy – 1011001

yx – 0011011

yxλx ~ 00110111011

Practice 17: palindromic strings.

- **operations** (e.g. string concatenation - Practice 18, p. 126)
- **algorithms** (e.g. BinarySearch - Practice 20, p. 131; check out Example #41, p. 130, for the definition of "middle".)

  Or my favorites, such as unix shell scripts. Here's one one might call "recurse", for applying an operations to all "ordinary" files:

```
#!/bin/sh
command=$1
files=`ls`
for i in $files
do
        if test -d $i
        then
            cd $i
            directory=`pwd`
            echo "changing directory to $directory..."
            recurse "$command"
            cd ..
        elif test -h $i
        then
            echo $i is a symbolic link: unchanged
else
            $command $i
        fi
done
```

2 Solving Recurrence Relations

Vocabulary:

- **linear recurrence relation:** $S(n)$ depends linearly on previous $S(r)$, $r < n$:

$$S(n) = f_1(n)S(n-1) + \cdots + f_k(n)S(n-k) + g(n)$$

if $g(n)=0$, homogeneous

Fibs : $F(n) = \underline{1} \cdot F(n-1) + \underline{1} \cdot F(n-2)$ 2nd order

constant coefficients

Facs : $F(n) = n\,F(n-1)$

— function of n, $f_1(n)$

The relation is called **homogeneous** if $g(n) = 0$. (Both Fibonacci and factorial are examples of homogeneous linear recurrence relations.)

- **first-order:** $S(n)$ depends only on $S(n-1)$, and not previous terms. (Factorial is first-order, while Fibonacci is second-order, depending on the two previous terms.)

- **constant coefficient:** In the linear recurrence relation, when the coefficients of previous terms are constants. (Fibonacci is constant coefficient; factorial is not.)

our objective:

- **closed-form solution**: $S(n)$ is given by a formula which is simply a function of n, rather than a recursive definition of itself. (Both Fibonacci and factorial have closed-form solutions.)

The author suggests an "expand, guess, verify" method for solving recurrence relations.

Example: The story of T

1. Practice 11, p. 121

$$T(1) = 1$$
$$T(n) = T(n-1) + 3$$

inhomogeneous

$g(n) = 3$

A few terms: $1, 4, 7, 10, 13, 16, \ldots$

2. Practice 19, p. 128: Here is the recurrence relation for Example 11, p. 121, in lisp:

```
(defun Tee(n)
  (if (integerp n)
      (cond
       ((>= n 2)
        (+ (Tee (- n 1)) 3)
        )
       ((= n 1)
        1
        )
       (t (print "Tilt! Only positive ints allowed..."))
       )
    (print "Tilt! Only positive ints allowed...")
    )
  )
> (tee 2)
4
> (mapcar #'tee (iseq 1 10))
(1 4 7 10 13 16 19 22 25 28)
```

3. Practice 21, p. 133

$$\boxed{T(n) = 3(n-1) + 1}$$

$T(1) = 1$ ✓

$T(2) = 4$

$T(3) = 7$

Assume $P(k)$:

$T(k) = 3(k-1)+1$.

$$T(k+1) = T(k) + 3$$
$$= \left(3(k-1)+1\right) + 3$$
$$= 3k + 1$$
$$= 3\left((k+1)-1\right) + 1$$

✓