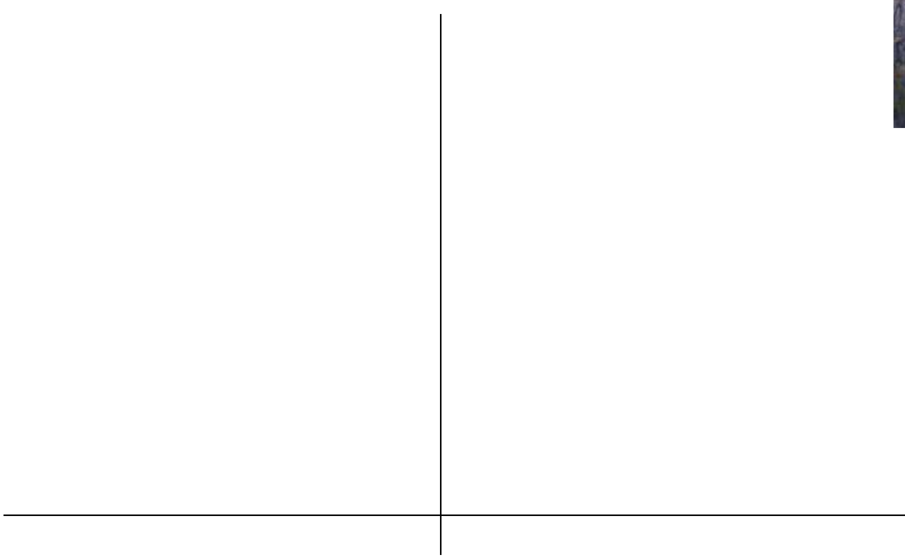
MAT122 Final (Fall 2007): Part II: applications

Name:

Directions: There are three problems: you are to do only one. Write "SKIP" plainly on the others. Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answer to each problem (e.g., put a box around it). **Good luck!**

Problem 1.

Ants, termites, and other critters build heaps of sand, clay, or other types of soil. These heaps tend to be conical (maybe even parabolic in shape). Imagine, then, that we are in the presence of a 3 meter tall clay termite mound, which was constructed from ground level. At its base, the mound is a meter and a half in diameter. Its two-dimensional vertical cross-section through the middle of the mound we model with a parabola (and we assume symmetry). The density of dry clay is $1089 kg/m^3$



a. (5pts) In the axes above, draw a picture of the situation, including a 3-d perspective. Place the origin at ground level, in the center of the termite mound, and make the z-direction the vertical direction.

b. (5pts) If you slice the mound **vertically** down the center, what is the area of a cross-section?

(Problem 1, cont.)

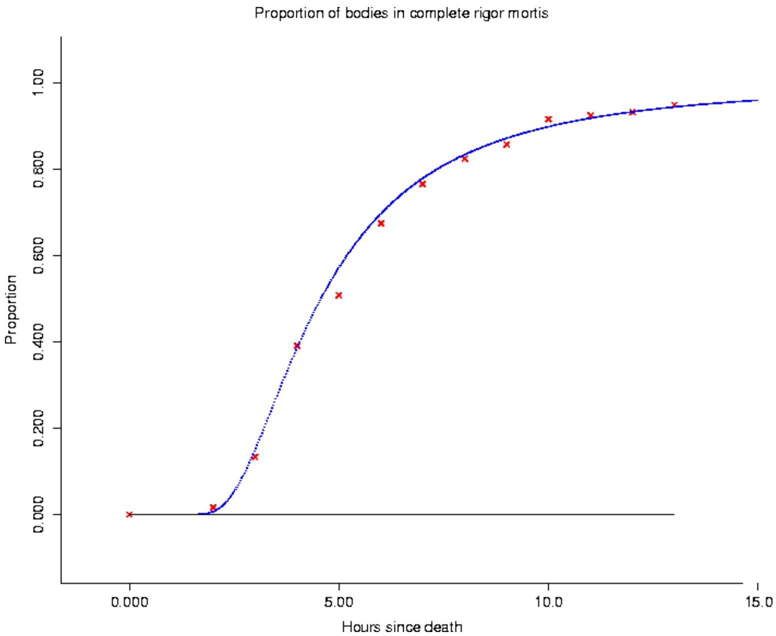
c. (5pts) If you slice the mound **horizontally** at a fixed height of z, what is the area of a cross-section?

d. (10pts) What is the work performed by the termites against gravity in lifting the clay from ground level to form the mound?

e. (10pts) What is the volume and mass of the mound?

Problem 2. When we die, our bodies become rigid (*rigor mortis* sets in). Niderkorn's (1872) observations on 113 bodies provides the main reference database for the development of *rigor mortis* and is commonly cited in textbooks. In preparation for this exam I fit a lovely model to this somewhat unlovely data, for the proportion $p(t)$ of bodies in *rigor mortis* after t hours. It is illustrated in the graph below: the model is

$$p(t) = e^{\left(-26.28/t^{2.39}\right)}$$



a. (10pts) You are called to the scene of an act of genocide, and find 100 bodies strewn over the landscape. 87 of the bodies show complete *rigor mortis*. In the space above, find the time that the act occurred, as predicted by the model. (Show how you arrive at the answer – using the "solve" button on your calculator is a start, but not the end: you should be able to find the solution analytically – that is, by hand, without recourse to the calculator).

b. (5pts) Compute the derivative by hand, and carefully plot it (either below, or in the graph above):

(Problem 2, cont.)

c. (5pts) Compute the average proportion of bodies in *rigor mortis* in the time from 3 to 5 hours after death (write the integral, but you may use your calculator!).

d. (5pts) How do you interpret areas such as $\int_{t_1}^{t_2} p'(t)dt$?

e. (5pts) From the graph you can guess that the limit of $p(t)$ as $t \to \infty$ is 1: demonstrate that it is so.

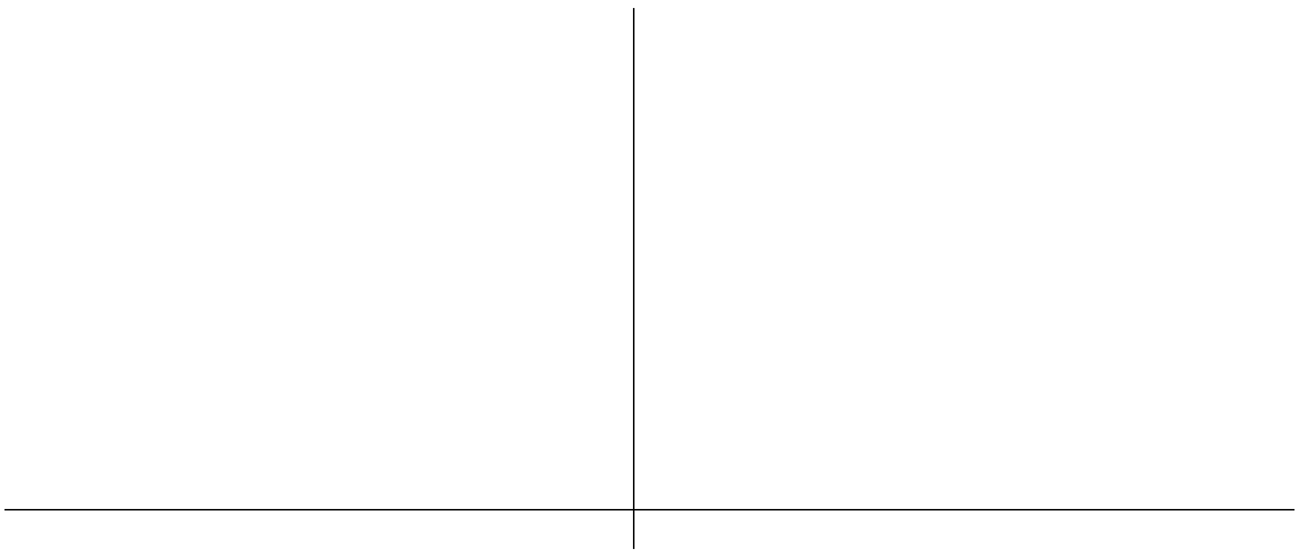
f. (10pts) How do we know that p is invertible, and what properties can you deduce about the inverse from the study we've done so far? (A graph would be a good way to summarize! I can think of at least five things that I know about the inverse....)

Problem 3. An important distribution in statistics is the standard normal distribution, associated with the function

$$\rho(x) = \frac{1}{\sqrt{2\pi}} e^{\left(\frac{-x^2}{2}\right)}$$

Here's an example (and admittedly silly) scenario which would be represented by ρ: a drunk takes a step left or right with equal probability. If we start the drunk off at the origin, and s/he can go only along a straight path (the x-axis – think perhaps of an alley – where will the drunk be after a fixed time? Well, equally likely to be found on the left or the right of the origin, obviously; more likely to be found near the origin than farther away. In fact, with proper scaling, ρ will give us the probability that the drunk is found in any portion of the x-axis.

a. (5pts) Plot $\rho(x)$ on the axis below. (Why is the distribution called the "bell curve"?)



b. (5pts) As I said, the probability of the drunk being found in any particular interval along the x-axis will be determined by ρ: more particularly, the probability P that the drunk's position x is between positions a and b is given by an area:

$$P(a \leq x \leq b) = \int_a^b \rho(x)dx$$

Unfortunately it is not possible for us to solve this integral analytically: but of course we can approximate it numerically.

Use your calculator to estimate the probability that the drunk is found between

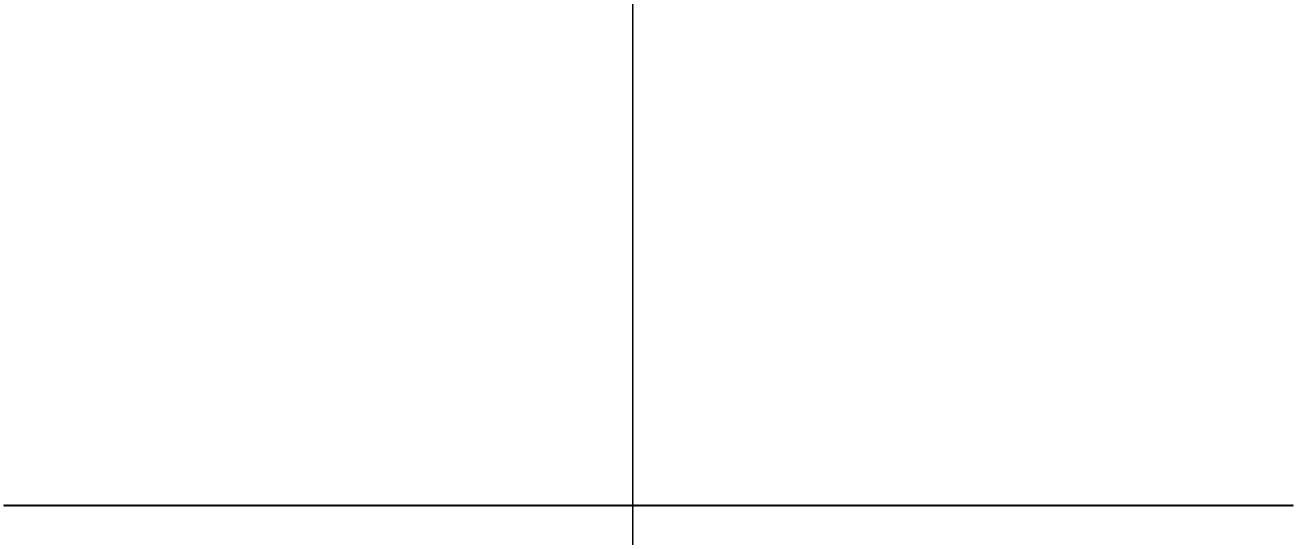
i. $x = -1$ and $x = 1$:

ii. $x = 1$ and $x = 2$:

iii. $x = -4$ and $x = 4$:

c. (5pts) Now really the drunk can stagger in two dimensions (ignoring falling down!;). It seems likely that the bell curve should be extended to the plane, by rotating it about the y-axis. Now probabilities will be given by volumes.

Plot the distribution that you get by rotating $\rho(x)$ about the y-axis:



d. (15pts) Set up and evaluate the appropriate integral that represents the probability that the drunk is found at a distance of

i. 1 unit from the origin in any direction: do the integral **by hand** via u-substitution.

ii. 4 units from the origin in any direction (you may use your calculator).

iii. Between 1 and 2 units from the origin in any direction (you may use your calculator).

e. (5pts) It turns out that we're **almost** right: we're off by a multiplicative factor. Given that probabilities are always between 0 and 1, estimate the multiplicative factor by which we must multiply the integrals to get true probabilities. Explain your reasoning!