Section 1.1: Statements, Symbolic Representations, and Tautologies

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Abstract

We encounter the elements of logic: statements, connectives, tautologies, contradictions, etc., and create wffs ("whiffs") from these basic elements. An algorithm for detecting tautologies in the form of implications is described.

Note: dual labelled exercises refer to 5th/6th edition numbers. Hence #26/29 refers to problem 26 in the 5th edition, and 29 in the 6th edition.

- **Statement/proposition**: a sentence possessing truth value (T or F).

Exercise #1

	Statement?	Truth value?
The moon is made of green cheese	Y	F
He is certainly a tall man. (free variable)	N	
Two is a prime number.	Y	T
Will the game be over soon? (Questions have no truth value....)	N	
Next year interest rates will rise.	"Y"	??
" " " " " fall.	"Y"	??

but which interest rates? Isn't that "free"?

- **Logical connectives** join statements into **formulas**, or compound statements:
  - conjunction (symbolized by $\wedge$, "and")

Context is important!

Language is confusing!

$A \leftrightarrow B \iff (A \rightarrow B) \wedge (B \rightarrow A)$

A	B	$A \wedge B$	$A \vee B$	$A \longrightarrow B$	A'	B'	$A \leftrightarrow B$	$(A \rightarrow B)'$	$A \wedge B'$
T	T	T	T	T	F	F	T	F	F
T	F	F	T	F	F	T	F	T	T
F	T	F	T	T	T	F	F	F	F
F	F	F	F	T	T	T	T	F	F

– disjunction (symbolized by $\vee$, "or")

– implication (symbolized by $\longrightarrow$: *(does its table seem weird to you? It's by convention!)* In the implication $A \longrightarrow B$, A is the **antecedent**, and B is the **consequent**. Some English equivalents to implication are given in Table 1.5.

Exercise #4

a. sufficient water → healthy growth

True ⇒ True? If so, the implication is okay

c. errors → modification

T ⇒ T (modifications were made)

Could have modification w/o error, so mod → error falsifiable,

Implication plays an especially important role among connectives, so learn it well!

– equivalence (symbolized by $\longleftrightarrow$, "if and only if")

– negation (symbolized by $'$, "not" – *unary*)

Note: These connectives are not independent - some of these may be derived from the others (Exercise #29/33 shows that conjunction and negation suffice to write the others, for example).

Exercise #6/7abc Negate this:

6a. food good → service excellent

$A \wedge (B)'$

$(A \rightarrow B)' \iff A \wedge B'$

b. food good ∨ service excellent

(food good)' ∧ (service excellent)'

c. (food good ∧ service excellent) ∨ price high

Example (more interesting, and demonstrating that context is important for a statement's truth value): The dilemma of Protagoras and Eualthus

- **Well-formed formula** (wff - "whiff") is a compound statement made up of statements, logical connectives, and other wffs *What makes one well-formed?*
  - **Order of precedence**:
    * parentheses
    * $'$
    * conjunction, disjunction
    * implication
    * equivalence

    Order of precedence helps us to simplify our lives: hence,

$$A \wedge B \longrightarrow C \text{ means } (A \wedge B) \longrightarrow C$$

  - **main connective** (last to be applied)

- **Truth table** for a wff with n statement letters: 2^n rows

  Example: the table for implication above, which is a binary (2 statement letter) logical connective. Hence there are $2^2 = 4$ rows.

- **tautology**: wff which is always true (represented by 1).

- **contradiction**: wff which is always false (represented by 0).

- **equivalent wffs**: wffs A and B are equivalent, $A \iff B$, if the wff

$$A \longleftrightarrow B$$

  is a tautology. (*How can we prove that?*)

  Some tautological equivalences:

1a. $A \vee B \iff B \vee A$	1b. $A \wedge B \iff B \wedge A$	Commutative
2a. $(A \vee B) \vee C \iff A \vee (B \vee C)$	2b. $(A \wedge B) \wedge C \iff A \wedge (B \wedge C)$	Associative
3a. $A \vee (B \wedge C) \iff (A \vee B) \wedge (A \vee C)$	3b. $A \wedge (B \vee C) \iff (A \wedge B) \vee (A \wedge C)$	Distributive
4a. $A \vee 0 \iff A$	4b. $A \wedge 1 \iff A$	Identity
5a. $A \vee A' \iff 1$	5b. $A \wedge A' \iff 0$	Complement

Equivalent wffs will be useful when we are proving arguments, and want to replace complex wffs with simpler ones.

- **De Morgan's Laws** are two specific examples of equivalent wffs:
  - $(A \vee B)' \iff A' \wedge B'$
  - $(A \wedge B)' \iff A' \vee B'$

Hence we claim that $(A \vee B)' \longleftrightarrow (A' \wedge B')$ is a tautology.

Exercise #17/20e

Demonstrate that $(A \vee B)' \leftrightarrow A' \wedge B'$

A	B	$A \vee B$	$(A \vee B)'$	$A' \wedge B'$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

Notice that the two formulas of De Morgan's Laws appear analogous ("dual"). In fact, one is the negation of the other.

Question: How so?

Exercise #24/27

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If [not((Value1 < Value2) or odd(Number))
   or (not(Value1 < Value2) and odd(Number))] then
      statement1
   else
      statement2
   end if
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(Value1 < Value2) = A, odd(Number) = B

$(A \vee B)' \vee (A' \wedge B)$

$\iff (A' \wedge B') \vee (A' \wedge B)$

$\iff A' \wedge (B' \vee B)$

$\iff A' \wedge 1$

$\iff A'$

- **Algorithm**: a set of instructions that can be mechanically executed in a finite amount of time in order to solve some problem.

  Often written out in **pseudocode**, the author provides us an example: the algorithm TautologyTest is useful for whether or not an implication (that is, a wff where the main connective is implication) is, in fact, always true (a tautology). She proceeds by contradiction (one proof technique we'll study further in Chapter 2): assume that the implication $P \to Q$ is false. Then P must be true, and Q false (the only scenario which makes an implication false).

  Exercise 26/29: b,d

  26 b: $[(A \to B) \wedge A] \to B$

  In pursuit of a contradiction, assume

① $[(A \to B) \wedge A]$	T	
② B	F	
③ $\therefore A \to B$	T	} based on line ①
④ A	T	
⑤ $\therefore B$	T	} based on line ③

  Building a truth table for the implication also constitutes an algorithm to test to see if it is true, but, although the truth table algorithm may be more powerful (as more general, working for all would-be tautologies), TautologyTest may be faster when applied to an implication.

  We've arrived at a contradiction, so our original assertion are wrong.
  This is in fact a tautology (always T)

#31 Prove that there are compound statements that are not equivalent to any statement using only the connectives $\to$ and $\vee$.

Challenge: Given A & B, True both. Construct a wff that is false based on

A, B, & the other connectives, then

→ & ∨ can't reproduce this wff.