Section 1.2: Propositional Logic

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Abstract

Now we're going to use the tools of formal logic to reach logical conclusions based on wffs formed by given statements. This is the domain of propositional logic.
Note: dual labelled exercises and page numbers refer to 5th/6th edition numbers, respectively.

- **Propositional wff**: represent some sort of argument, to be tested, or proven, by **propositional logic**.

- **valid arguments**, e.g.
$$P_1 \wedge P_2 \wedge \ldots \wedge P_n \rightarrow Q$$
have **hypotheses** (we suppose that the P_i are true), and a **conclusion** (Q). To be *valid*, this argument must be a tautology (always true). To be an *argument*, Q must not be identically true (i.e. a fact, in which case the hypotheses would be irrelevant!).

- **Proof Sequence**: a sequence of wffs in which every wff is a hypothesis or the result of applying the formal system's derivation rules (truth-preserving rules) in sequence.

  Objective: to reach the conclusion Q from the hypotheses P_1, P_2, ..., P_n.

- Types of derivation rules:
  - **Equivalence rules** (see Table 1.12, p. 23/24): we can substitute equivalent wffs in a proof sequence. One way of showing that two wffs are equivalent is via their truth tables.
    * commutative
    * associative
    * De Morgan's laws
    * implication $P \rightarrow Q \iff P' \vee Q$
    * double negation

Implication seems somewhat unusual, but it is suggested by Exercise 6/7a, section 1.1.

You're asked to prove it in Practice 9, p. 23/24. That is, prove that

$$\boxed{P \to Q \longleftrightarrow P' \vee Q}$$

is a tautology. How would you do it?

- **Inference rules**: from given hypotheses, we can deduce certain conclusions (see Table 1.13, p. 24/25)
  - **modus ponens:** If Q follows from P, and P is true, then so is Q. $P \to Q, P \Rightarrow Q$
  - **modus tollens:** If Q follows from P, and Q is false, then so is P. $P \to Q, Q' \Rightarrow P'$
  - **conjunction:** If Q is true, and P is true, then they're both true together. $P, Q \Rightarrow P \wedge Q$
  - **simplification:** If both Q and P are true, then they're each true separately. $P \wedge Q \Rightarrow P, Q$
  - **addition:** If P is true, then either P or Q is true. $P \Rightarrow P \vee Q$

Practice 10, p. 24/26. Also give step 4!

1. $(A \wedge B') \to C$ hyp
2. C' hyp
3. $(A \wedge B')'$ 1, 2, mt
4. $A' \vee (B')'$ de Morgan

For a more elaborate example, let's look at #27/29, p. 32/33, which shows that one can prove anything if one introduces a contradiction (e.g. the mensa quiz). Also called an **inconsistency**.

$P \wedge P' \rightarrow Q$

1.	P	hyp
2.	P'	hyp
3.	$P \vee Q$	1, add
4.	$P' \vee Q$	2, add
5.	$(P \vee Q) \wedge (P' \vee Q)$	3,4, con
6.	$(Q \vee P) \wedge (Q \vee P')$	5, comm
7.	$P \rightarrow Q$	4, imp
8.	Q	1,7 mp

– The difference between equivalence rules and inference rules is that equivalence rules are bi-directional (work both ways), whereas some inference rules are uni-directional (work in only one direction - this is what inference is all about: from this we can infer that, but we cannot necessarily infer this from that!).

Notice that in the table 1.14 (p. 31/33) some rules appear twice: two uni-directionals can make a bi-directional!

Note for your homework: you are not allowed to invoke the rule that you are trying to prove! Notice that the entries in this table are followed by exercise numbers: it is in those exercises that the results are obtained!

– **Deduction method**: if we seek to prove an implication, we can simply add the hypothesis of this conclusion implication to the hypothesis of the argument, and prove the conclusion of the remaining implication:

$$P_1 \wedge P_2 \wedge \ldots \wedge P_n \rightarrow (R \rightarrow S)$$

can be replaced by

$$P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge R \rightarrow S$$

If you're interested in seeing why this rule works, you might try #45/49, p. 33/34, but think of it this way: we're interested in assuming that all the P_i are true, and see if we can deduce the implication $R \rightarrow S$. If R is false, then the implication is true. The only potentially problematic case is where R is true, and S is false. Then what we want to know is: given that

$$P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge R$$

are true, is S true?

Exercise #32/34, p. 32/33

$(A' \to B') \wedge (A \to C) \to (B \to C)$

1. $A' \to B'$ hyp
2. $A \to C$ hyp
3. B hyp, ded method
4. $(B')' \to (A')'$ 1, cont
5. $B \to A$ dn twice
6. A 3,5 mp
7. C 2,6 mp

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1. $A' \to B'$ hyp
2. $A \to C$ hyp
3. $B \to A$ 1, cont + dn
4. $B \to C$ 2, 3, hs

– **Hypothetical syllogism**:

$$(P \to Q) \wedge (Q \to R) \to (P \to R)$$

(and see a whole long list of rules in Table 1.14). This rule might be referred to as **transitivity**.

A new rule is created each time we prove an argument; but we don't want to create so many rules that we keel over under their weight! Keep just a few rules in view, and learn how to use them well....

- Our goal may well be to turn a "real argument" into a symbolic one. This allows us to test whether the argument is sound (that is, that the conclusion follows from the hypotheses).

Exercise #39/41, p. 32/34.

If Jane is more popular then she will be elected. If Jane is more popular then Craig will resign. $\therefore$ If[4] Jane is more popular then she will be elected & Craig will resign.

$(J \to E) \wedge (J \to C) \quad \therefore \to [J \to E \wedge C]$

1. $J \to E$	hyp		4. E	1, 3	mp
2. $J \to C$	hyp		5. C	2, 3	mp
3. J	hyp, ded method		6. $E \wedge C$	4, 5	conj.

- The propositional logic system is complete and correct:
  - **complete**: every valid argument is provable.
  - **correct**: **only** a valid argument is provable.

  The derivation rules are truth-preserving, so correctness is pretty clear; completeness is not! How can we tell if we can prove every valid argument?!