Section 1.3: Quantifiers, Predicates, and Validity

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Abstract

We now throw variables into the mix, and investigate wffs which describe properties of the domains of those variables in given "interpretations." We still test their truth values, either for the specific domain in question, or even in all domains (validity).

1 Predicates and quantifiers

- **quantifier**: tells how many objects in a given domain have a certain property.

  Examples:

  - **universal quantifier** - $\forall$ - "for all", "for every", "for any"
  - **existential quantifier** - $\exists$ - "there exists", "for at least one", "for some"

  Have you encountered these quantifiers before, in other courses?

- **predicate**: a property of a variable (e.g. "x is prime"), generally containing one or more variables (and perhaps some constants).

  We combine the quantifiers and predicates to create expressions (wffs) such as

  $$(\forall x)P(x)$$

  which we then must *interpret*. For example, this might be said in the context of the integers, with $P(x)$ standing for "x is prime". (So this wff would be false in this context.)

  There is nothing special about the variable x, so this wff is the same as $(\forall y)P(y)$, $(\forall z)P(z)$, etc. We say that x is a *dummy* variable.

  Predicates may have any number of variables in them: the example above is a *unary* predicate, with only a single variable.

- Truth value hence now depends on the **Interpretation** of an expression:

– **domain of interpretation** - non-empty set to which the predicate expression is applied
– assignment of a property of the objects to each predicate in the expression
– assignment of particular objects to each constant symbol in the expression

We start with something abstract, and replace it with concrete instances in a given context.

Example: Ex. 2(e, f), p. 42/43

$(\forall x)(\forall y)(Q(x,y) \vee P(x,y))$

2e. $(\forall x)[(\forall y)(x<y \vee y<x)]$

false : x could equal y.

Interpretation :

① Domain is $\mathbb{Z}$

② $Q(x,y)$ is "$x<y$"
$P(x,y)$ is "$y<x$"

③ no statement letters

2f. $(\forall x)[x<0 \rightarrow (\exists y)(y>0 \wedge x+y=0)]$

true : choose $y=-x$

Example: Ex. 4/5(b, d), p. 42/44

4b. $(\forall x)(\forall y)[P(x,y) \rightarrow P(y,x)]$

Domain : $\mathbb{Z}$ $P(x,y)$: "$x=y$"

True in this ~~~~ interpretation

Same domain : $P(x,y)$: "$x>y$"

False in this one.

In example 4/5b above, the predicate $P(x, y) \longleftrightarrow P(y, x)$ is an example of a **binary** predicate.

- **scope** of a quantifier: the part of an expression to which the quantifier applies; indicated with parentheses or brackets (but these may be neglected if the scope is clear).

- **free variable**: a variable in a predicate wff outside the scope of a quantifier involving that variable.

Example: Ex. 5/6(c), p. 42/44

y is free in this predicate wff.

5e. $$(\exists x)\underbrace{\left[(\forall y)\overbrace{P(x,y)}^{\text{scope of } \forall y} \land \overbrace{Q(x,y)}^{y}\right]}_{\text{scope of } y}$$

2 Translation of English statements into predicate wffs

As noted before, this can be a very tricky business, but an important one. As is often the case, the process of translation does not result in a unique expression: there may be several different ways to do the same job.

Our author encourages us to remember that

- typically $\exists$ and $\wedge$ go together, whereas
- typically $\forall$ and $\longrightarrow$ go together.

Also, a single English statement may be given by numerous wffs.

Example: Ex. 9/14(e, g), p. 43/46

9e. Only judges admire judges

$J(x)$ - x is a judge

$A(x,y)$ - x admires y

9g: $L(x)$ - x is a lawyer

$W(x)$ - x is a woman

"Some women admire no lawyer."

$$\cancel{(\exists x)(\forall y)[W(x) \land L(y) \to A(x,y)']}$$

doesn't work!

$(\forall x)(\forall y)[J(x) \wedge A(y,x) \to J(y)]$

Example: Ex. 11/16(a-d), p. 44/47

a. $(\forall x)(\forall y)[B(x) \wedge F(y) \to L(x,y)]$

$(\forall x)[B(x) \to (\forall y)(F(y) \to L(x,y))]$

b. $(\exists x)(\forall y)[B(x) \wedge F(y) \to L(x,y)]$

$(\exists x)[B(x) \wedge (\forall y)[F(y) \to L(x,y)]]$

c. $(\forall x)[B(x) \to (\exists y)[F(y) \wedge L(x,y)]]$

9, cont.

$(\exists x)[W(x) \wedge$

$(\forall y)[L(y) \to A(x,y)']]$

(see bottom for more)

- **Negation** of predicate wffs: some cases are standard, e.g.
  - The negation of "Every x has property A." is "There is an x which doesn't have property A."
  $$[(\forall x)A(x)]' \iff (\exists x)[A(x)]'$$
  - The negation of "There is an x which has property A." is "No x has property A."
  $$[(\exists x)A(x)]' \iff (\forall x)[A(x)]'$$

In general, English makes negation kind of tricky. Watch your step!

Example: Ex. 14/19(c,d), p. 44/48

14c. $((\forall x)[\text{Tall} \wedge \text{Thin}])'$

$(\exists x)[\text{Tall} \wedge \text{Thin}]' \iff (\exists x)[\text{Tall}' \vee \text{Thin}']$

d. (Some pictures are old or faded)'

$(\forall x)[\text{old or faded}]' \iff (\forall x)\ \text{old}' \wedge \text{faded}'$

3 Validity

The truth value of a predicate wff depends on the interpretation, but there are some for which the wff is true independent of the interpretation. These are called **valid** predicate wffs (the analogue of tautology for propositional wffs).

Whereas we can check the "validity" of a propositional wff (just check the truth table to see if it's a tautology), there is no general check for the validity of a predicate wff, since it depends on context. In spite of that, there are some valid predicate wffs (context free truth!), as demonstrated in the text:

$$(\forall x)P(x) \to (\exists x)P(x)$$
$$(\forall x)P(x) \to P(a)$$
$$P(x) \to (Q(x) \to P(x))$$

Example: Ex. 18/24(d,e), p. 45/49

d. $A(a) \to (x)A(x)$

e. $(\forall x)[A(x) \to B(x)] \to [(\forall x)A(x) \to (\forall x)B(x)]$

More on 9g: The first one we tried didn't work, & neither does this.

$(\exists x)[(\forall y)[W(x) \wedge A(x,y) \to L(y)']]$

No - anything in the domain that's not a woman makes this true. Same with this one,

$(\exists x)[(\forall y)[W(x) \wedge L(y) \to A(x,y)']]$

and same with our initial version:

"Some women admire no lawyer."

~~$(\exists x)(\forall y)[W(x) \wedge L(y) \rightarrow A(x,y)']$~~

All of these fail to capture the fact that we are asserting the existence of a woman who admires no lawyers.

$$(\exists x)[W(x) \wedge \dots]$$

I got back on track at the board only for a moment when I looked at one of these & said "A man makes it true!" But someone threw me off my game by asserting that "Yes, but we've said there's a woman, too...."

That's correct, but the fact of the matter is that all three of these wffs are true even if no woman admires only lawyers!

Because a man will make it true, regardless.

And that is a bad thing....

10.f If there is a Corvette that is slower than a Ferrari, then all Corvettes are slower than all Ferraris.

$$(\exists x)[C(x) \wedge (\exists y)[F(y) \wedge S(x,y)]] \longrightarrow$$

$$(\forall x)(\forall y)[C(x) \wedge F(y) \rightarrow S(x,y)]$$

20/26 c

$$(\forall x)A(x) \rightarrow ((\exists x)(A(x))')'$$