Section 2.2: Induction

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Abstract

In this section we investigate a powerful form of proof called **induction**. This is useful for demonstrating that a property, call it $P(n)$, holds for all integers n greater than or equal to 1.

Actually, the "1" above is not essential: any "base integer" will do (like 0, for example: it really only matters that there be a "ground floor", or "anchor").

1 Induction

Induction is a very beautiful and somewhat subtle method of proof: the idea is that we want to demonstrate a property associated with natural numbers (or a subset of the natural numbers). As a typical example, consider a theorem of the following type:

Prove that, for any natural number n, $1+2+3+\ldots+n=\frac{n(n+1)}{2}$ (Gauss's theorem, stated when he was seven or so).

An induction proof goes something like this:

- We'll show that it's true for the first case (usually $k=1$, called the base case). While the first case is often $k=1$, this isn't mandatory: we simply need to be sure that there is a first case for which the property is true. $k=0$ is another popular choice....

- Then we'll show that, if the property is true for the k^{th} case, then it's true for the $(k+1)^{th}$ case (the inductive step).

- Then we'll put them together: if it's true for 1, then it's true for 2; if it's true for 2, then it's true for 3; .... "to infinity, and beyond!" Or up the ladder, as our author would say.

Imagine dominoes falling. That's what it's like.

The most commonly used form of the principle of induction is expressed as follows:

First Principle of Mathematical Induction:

$$\left.\begin{array}{l} 1.\ P(1) \text{ is true} \\ 2.\ (\forall k)[P(k) \text{ true } \rightarrow P(k+1) \text{ true }] \end{array}\right\} \rightarrow P(n) \text{ true for all positive integers } n$$

or, more succinctly,

$$P(1) \wedge (\forall k)[P(k) \rightarrow P(k+1)] \rightarrow (\forall n)P(n)$$

where the domain of the interpretation is the natural numbers. This is just *modus ponens* applied over and over again. Put *modus ponens* into an infinite loop, because we want it to run off to infinity! This might be the first infinite loop you've ever liked....
Vocabulary:

- **inductive hypothesis**: $P(k)$
- **basis step** (base case, anchor): establish $P(1)$ ✓
- **inductive step** (implication): $P(k) \rightarrow P(k+1)$ ✓

Example: (Practice 7, or "Gauss's theorem") Prove that, for any natural number n, $1+2+3+\ldots+n = \frac{n(n+1)}{2}$.

(1) Anchor: prove true for $n=1$

$P(n)$: $1+2+3+\ldots+n = \frac{n(n+1)}{2}$

$P(1)$: $1 = \frac{1(1+1)}{2}$ ✓

(2) Implication: $P(k) \rightarrow P(k+1)$ (prove this!)

$P(k)$: $1+\ldots+k = \frac{k(k+1)}{2}$ (assume)

$P(k+1)$: $1+\ldots+k+(k+1) = \frac{(k+1)((k+1)+1)}{2}$

1. $P(k)$ hyp
2. $1+\ldots+k = \frac{k(k+1)}{2}$

3. $\underbrace{1+\dots+k}_{\frac{k(k+1)}{2} \text{ by hyp!}} + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)+2(k+1)}{2}$

$= \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}$

4. $P(k+1)$ ✓ | ∴ The theorem is proven by induction.

Example: Exercise 34, p. 106/114: Prove that $\underbrace{2^{n-1} \leq n!}_{P(n)}$ for $n \geq 1$.

① Anchor

$P(1): \ 2^{1-1} \leq 1! \iff 1 \leq 1$ ✓

② Implication: Assume $P(k)$: $2^{k-1} \leq k!$; show

$P(k+1)$: $\underbrace{2^{(k+1)-1}}_{2^k} \leq (k+1)! = (k+1)\cdot k!$

$2^k = 2^{k-1}\cdot 2 \leq k!\cdot 2 \leq k!\,(k+1) = (k+1)!$ ✓

∴ The theorem holds by mathematical induction.

A second (and seemingly more powerful) form of induction is given by the **Second Principle of Mathematical Induction**:

$$\left.\begin{array}{l} 1.\ P(1) \text{ is true} \\ 2.\ (\forall k)[P(r) \text{ true for all } r, \\ \qquad 1 \leq r \leq k \to P(k+1) \text{ true }] \end{array}\right\} \to P(n) \text{ true for all positive integers } n$$

This principle is useful when we cannot deduce $P(k+1)$ from $P(k)$ (for k alone), but we can deduce $P(k+1)$ from all preceeding integers, beginning at the base case.

Example: Exercise 64/66b, p. 109/116.

n-sided polygon (closed)

$P(n)$: sum of the interior angles is $(n-2)\cdot 180$

Claim holds for $n \geq 3$

① Anchor: $P(3)$ sum of interior angles of a triangle is $180 = (3-2)\cdot 180$ ✓

② Implication: Assume $P(r)$ for $3 \leq r \leq k$, & show $P(k+1)$.

sum of the interior angles of the m- & p-gons is the same as the sum for the k+1-gon, which is therefore $A = (m-2)\cdot 180 + (p-2)\cdot 180$

by the inductive hypothesis.

$m + p = k + 1 + 2$

$\therefore A = (m+p-4)\cdot 180 = (k+1+2-4)\cdot 180 = (k+1-2)\cdot 180$ ✓

$P(k+1)$

Two checks! $\therefore$ the result holds by induction.

Each of these two principles is equivalent to the **Principle of Well-Ordering**, which states that every collection of positive integers that contains any members at all has a smallest member.

Example: Prove that the first principle of induction implies well-ordering.

A Final Example: The prisoner's last request (finite backwards induction!)