Section 3.1: Sets

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Abstract

This section, the only section we consider from Chapter 3, simply gives us some basic vocabulary and notions of sets that we will need when we get to Boolean algebras later. It also includes some really interesting examples of ideas from set theory (e.g. different sizes of infinite sets – did you know that infinity comes in infinitely many different sizes?).

1 Notation

A set (call it A) is loosely a collection of objects.

Capital letters denote sets, and $\in$ denotes inclusion in a set, so that $x \in A$ means that x is a member (or element) of a set, and $x \notin A$ means that x isn't a member.

Sets are unordered: the order in which the elements are listed is unimportant.

We use predicate logic to determine when two sets are **equal**:

$$A = B \iff (\forall x)[(x \in A \to x \in B) \wedge (x \in B \to x \in A)]$$

The notation for a set whose elements are characterized by possessing property P is

$$S = \{x | P(x)\}$$

and is read "S is the set of all x such that $P(x)$"

One curiously useful set is the **empty** set, denoted $\oslash$ or $\{\}$.
Some important sets of numbers:

$\mathbb{N}$	The natural numbers
$\mathbb{Z}$	The integers
$\mathbb{Q}$	The rational numbers
$\mathbb{I}$	The irrational numbers
$\mathbb{R}$	The real numbers
$\mathbb{C}$	The complex numbers

Example: Practice 3, p. 165/189. Describe each set:

1. $A = \{x | x \in \mathbb{N} \text{ and } (\forall y)(y \in \{2,3,4,5\} \rightarrow x \geq y)\}$

   $= \{5, 6, 7, 8, 9, \ldots\}$

2. $B = \{x | (\exists y)(\exists z)(y \in \{1,2\} \text{ and } z \in \{2,3\} \text{ and } x = y + z)\}$

   $= \{3, 4, 5\}$

2 Relationships between Sets

A is a **subset** of B, denoted $A \subseteq B$, if

$$(\forall x)(x \in A \rightarrow x \in B)$$

and A is a **proper subset** of B, denoted $A \subset B$, if

$$(\forall x)(x \in A \rightarrow x \in B) \wedge (\exists x)(x \notin A \wedge x \in B)$$

Example: Practice 6, p. 166/190

C - even natural numbers

a. T   b. T

c. F   d. T

e. T   f. F

g. F   h. T

i. T   j. F

k. F   l. T

Theorem:

$$A = B \iff (A \subseteq B \wedge B \subseteq A)$$

3 Sets of Sets

Power Set: Given set S, the power set of S, denoted $\wp(S)$, is the set of all subsets of S. (Note that S and $\emptyset$ are elements of the power set of S.)

Example: How big is the power set of a given set? (Practice 8 and 9, p. 168/191)

4 Binary and Unary Operations

We can create **ordered pairs** of elements of a set. From $A = \{1, 3, 4\}$ we can create the ordered pairs $(1, 3)$ and $(3, 3)$, for example. Now the order of the elements is important!

Question: How many distinctly different ordered pairs are there if we have a set with n elements?

Definition: $\circ$ is a **binary operation** on a set S if for every ordered pair (x, y) of elements of S, $x \circ y$ exists, is unique, and is a member of S.

Definition: $\circ$ is **well-defined** if $x \circ y$ exists and is unique.

Definition: $\circ$ is **closed** if $x \circ y \in S$.

Three ways to fail to be a binary operation on S:

1. there are pairs for which $x \circ y$ fails to exist;
2. there are pairs for which $x \circ y$ gives multiple different results;
3. there are pairs for which $x \circ y$ doesn't belong to S.

Definition: a **unary operation** on a set S associates with every element x of S a unique element of S.

Example: Practice 12, p. 170/193

5 Operations on Sets

Given a set S of elements of interest (the **universal set**), we may want to operate on various subsets of S (that is, elements of $\wp(S)$). For example,

Definition: Let $A, B \in \wp(S)$. The **union** of A and B, denoted $A \cup B$, is given by $\{x | x \in A \vee x \in B\}$. The **intersection** of A and B, denoted $A \cap B$, is given by $\{x | x \in A \wedge x \in B\}$.

These are examples of binary operations on the set of power sets of a set.

¡A HREF=”../venn/VennGame.html”¿ **Venn Diagrams**

¡/A¿ are useful tools for considering the notions of union and intersection. The diagrams in Figures 3.1 and 3.2 (p. 171/195) illustrate these notions “pictorially”.

Definition: For a set $A \in \wp(S)$, the **complement** of A, denoted A', is $\{x | x \in S \wedge x \notin A\}$.

Example: Practice 14, p. 171/195: illustrate A' using a Venn Diagram.

Definition: For set $A, B \in \wp(S)$, the **set-difference** of A and B, denoted $A - B$, is given by $\{x | x \in A \wedge x \notin B\}$.

Definition: For set $A, B \in \wp(S)$, the **Cartesian product (cross product)** of A and B, denoted $A\ X\ B$, is the set of all ordered pairs, and is given by

$$A\ X\ B = \{(x, y) | x \in A \wedge y \in B\}.$$

Example: Practice 15, p. 172/195: illustrate $A - B$ using a Venn Diagram.

6 Set Identities

We will encounter the following “Set identities” later in the context of “Boolean algebras”:

1*a*. $A \cup B = B \cup A$	1*b*. $A \cap B = B \cap A$	commutative prop
2*a*. $(A \cup B) \cup C = A \cup (B \cup C)$	2*b*. $(A \cap B) \cap C = A \cap (B \cap C)$	associative proper
3*a*. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	3*b*. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	distributive prope
4*a*. $A \cup \oslash = A$	4*b*. $A \cap S = A$	identity property
5*a*. $A \cup A' = S$	5*b*. $A \cap A' = \oslash$	complement prope

Notice the “dual” nature of the properties: it seems that the operations of $\cup$ and $\cap$ have a lot in common!

Question: What correspondence do you observe between these identities and those of wffs with the logical connective $\wedge$ and $\vee$?

7 Countable and Uncountable Sets

As an interesting application of set theory, we will now demonstrate that there are various sizes of infinity!

The natural numbers comprise the smallest infinity, a **denumerable** or **countable** infinity.

We prove that two sets are of equal size (even if infinite!) by creating a **one-to-one correspondence** between the two sets. If such a correspondence exists, then the two sets have the same size.

Theorem: the rational numbers are denumerable.

Theorem: the real numbers are **not** denumerable.

Theorem: the power set of a set S is always larger than S (punch line: there is always a bigger infinity than the one you already have).

Proof: By contradiction. Consider $f : S \longrightarrow \wp(S)$ a one-to-one correspondence between S and $\wp(S)$. That is, every set of $\wp(S)$ is represented by an element of S. But

$$A = \{x \in S | x \notin f(x)\} \notin f(S),$$

and yet $A \in \wp(S)$. This is a contradiction.