

Section 1.2: Row Reduction and Echelon Forms

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Abstract

In this section we consider an algorithm (Gaussian elimination with partial pivoting) for reducing a system of equations to reduced echelon form, which is unique and allows us to write the solution set of a linear system. The solution may contain free parameters, meaning that there are infinitely many solutions.

Definition: A rectangular matrix is in **row echelon form** if it has the following three properties:

1. zero rows are at the bottom;
2. Leading (non-zero) entries of ~~a~~ row move right as one descends the matrix; and
3. All entries below a row's leading entry are zero.

Example:

$$\begin{bmatrix} 1 & 4 & 9 & 7 \\ 0 & 2 & 6 & -1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

Definition: A matrix in echelon form is in **reduced row echelon form** if it has the additional two properties:

1. The leading entry in each non-zero row is 1, and
2. Each leading entry is the only non-zero entry in its column.

Example: The matrix above in reduced row echelon form is

$$\begin{bmatrix} 1 & 4 & 9 & 7 \\ 0 & 2 & 6 & -1 \\ 0 & 0 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & 1/3 \end{bmatrix}$$

Theorem: Each matrix is row-equivalent to one and only one reduced row echelon matrix.

Example: The 4 x 4 identity matrix I_4 :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: #2, p. 25

a. Reduced

b. echelon, not reduced

c. neither

d. E.F.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note: The uniqueness of the reduced row echelon matrix implies that the positions of the leading non-zero entries are fixed in stone for a given matrix. These positions are known as **pivot positions**. A **pivot column** of a matrix is one that contains a pivot position.

This leads to the following definitions:

- **basic variables:** variables corresponding to pivot positions
- **free variables:** variables other than basic variables

Gaussian elimination (using **partial pivoting**) for ^{reduced} row reduction:

1. Begin with the leftmost column. This is a pivot column, with pivot position at the top. Select largest entry (in terms of magnitude) in the pivot column, and interchange rows to move this entry into the pivot position.
2. Use row replacement to create zeros in all positions below the pivot.
3. Iterate, using the submatrix below the pivot row and to the right of the pivot column.
4. Beginning with the rightmost pivot and working left, create zeros above each pivot.
5. Scale all pivot rows by the pivot value, so that pivots are 1.

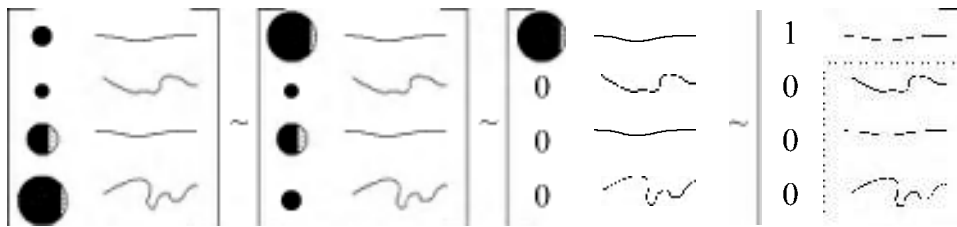


Figure 1: Here's my image of the partial pivoting process: notice that it's recursive, and we keep paring down the size of the matrix (dotted part in the last panel) until we run out of matrix to work on!

Example: Consider Exercise #12, p. 25. In this example, we have an augmented system of three equations in four unknowns. Hence, we know that we will not have enough information to have a unique solution: the question then is whether we are consistent, and how to express the solution set.

$$\begin{aligned}
 & \left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{array} \right] \\
 & \sim \left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & -4 & 8 & 12 \\ 0 & 0 & 1 & -2 & -3 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & -4 & 8 & 12 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \\
 & \sim \left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]
 \end{aligned}$$

The final matrix is annotated with blue and red circles around the pivot elements (1, 0, 0) and the right-hand side (5, -3, 0). Blue arrows point to the equations $x_1 - 7x_2 + 6x_4 = 5$ and $x_3 - 2x_4 = -3$. The word "basic" is written in blue, and "free" is written in red.

Our author suggests solving systems by pushing all the way to reduced echelon form. Here's his process for using row reduction to solve a linear system:

1. Write the augmented matrix of the system
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If so, continue; else stop.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix in reduced echelon form.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

$$\begin{aligned}
 & x_1 - 7x_2 + 6x_4 = 5 \\
 & x_3 - 2x_4 = -3
 \end{aligned}$$

$$\boxed{
 \begin{aligned}
 x_1 &= 5 + 7x_2 - 6x_4 \\
 x_3 &= -3 + 2x_4
 \end{aligned}
 }$$

Example: #7, p. 25 (what's the geometry like?) $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$

$$\sim \begin{bmatrix} 3 & 9 & 7 & 6 \\ 1 & 3 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 3 & 9 & 7 & 6 \\ 0 & 0 & 5/3 & 5 \end{bmatrix} \xrightarrow{-\frac{1}{3} \textcircled{1} + \textcircled{2}} \text{Observe - consistent so continue ...}$$

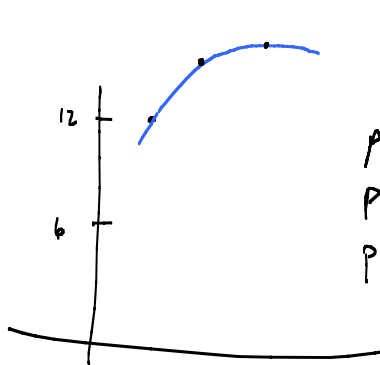
$$\sim \begin{bmatrix} 3 & 9 & 7 & 6 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & 9 & 0 & -15 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= -5 - 3x_2 \\ x_3 &= 3 \end{aligned}$$

General solutions of linear systems are hence given parametrically. If there is a free variable, then there are an infinite number of solutions. If there is a row having its pivot position in the rightmost column, then the system is inconsistent. This is the essence of Theorem 2, p. 24.

Additional problems: #15, #29, #30

#33



$$p(t) = a_0 + a_1 t + a_2 t^2$$

$$p(1) = 12 = a_0 + a_1 \cdot 1 + a_2 \cdot 1^2$$

$$p(2) = 15 = a_0 + a_1 \cdot 2 + a_2 \cdot 2^2$$

$$p(3) = 16 = a_0 + a_1 \cdot 3 + a_2 \cdot 3^2$$

$$\begin{bmatrix} 1 & 1 & 1 & 12 \\ 1 & 2 & 4 & 15 \\ 1 & 3 & 9 & 16 \end{bmatrix}$$