Section 1.3: Vector Equations

January 23, 2008

Abstract

Vectors provide a wonderful way for us to write systems of equations compactly. You should already be familiar with two-d and three-d vectors from calculus classes. We now want to extend notions from those spaces into n-dimensional space. For example, vector addition is carried out component-wise.

The interesting new concept introduced in this section is that of **span**: roughly, the span of a set of vectors $\{\mathbf{v}_1, ...\mathbf{v}_p\}$ is the subspace generated by linear combinations of the vectors $\mathbf{v}_i$. The span represents the set of vectors $\mathbf{b}$ that can be solutions of the system

$$a_1\mathbf{v}_1 + \ldots + a_p\mathbf{v}_p = \mathbf{b}$$

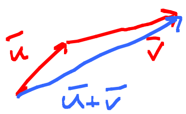
Definition: A **vector** is a matrix with only a single column ("column vector"). The entries are called the **components** of the vector.

- **zero vector**: the vector whose components are all 0: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- **one vector**: the vector whose components are all 1: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
- **scalar multiple** of a vector: a product of a constant ("**scalar**") and a vector, the operation being carried out component-wise: e.g.

$$\alpha\mathbf{v} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \alpha v_3 \end{bmatrix}$$

Note: here's a notational issue. Vectors will generally be in boldface (on the board I'll either underline them, or overline them, depending on my mood, time of day, and what I had for breakfast). The components of named vectors are generally written with the same name, only without bold/overline/underline, and with subscripts. Notice that the components $\{v_1, v_2, v_3\}$ of the vector $\mathbf{v}$ above are not at all the same as the vectors listed in the abstract, $\{\mathbf{v}_1, ...\mathbf{v}_p\}$. Components are (generally) numbers....

- **vector sum**: the vector created by adding two vectors, the sums being carried out component-wise. Naturally the vectors must have the same length....



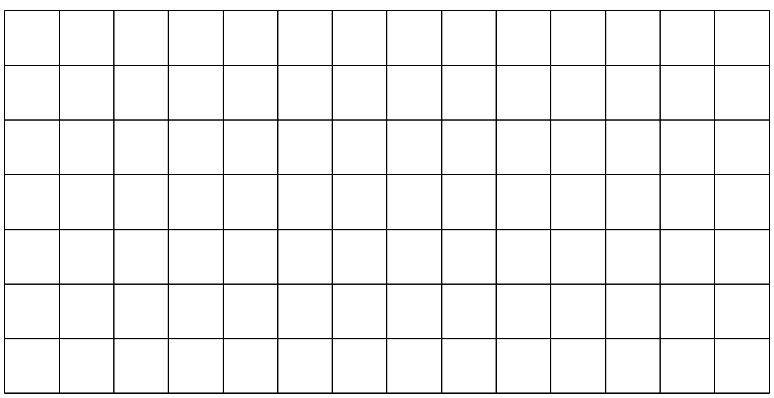
$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$$

Note: Geometrically, the sum of vectors can be found using the "parallelogram rule": the butt of vector $\mathbf{v}_2$ is placed at the tip of the vector $\mathbf{v}_1$, and the vector from the butt of $\mathbf{v}_1$ to the tip of $\mathbf{v}_2$ is the sum.

- **linear combination** of vectors: any sum of vectors scaled by coefficients. E.g.,

$$\alpha\mathbf{u} + \beta\mathbf{v} = \alpha \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \beta \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \alpha u_1 + \beta v_1 \\ \alpha u_2 + \beta v_2 \\ \alpha u_3 + \beta v_3 \end{bmatrix}$$

Example: #4, p. 37



- **span**: the span of a set of vectors $\{\mathbf{v}_1, ...\mathbf{v}_p\}$ is the subspace generated by linear combinations of the vectors $\mathbf{v}_i$. The span represents the set of vectors $\mathbf{b}$ that can be solutions of the system

$$a_1\mathbf{v}_1 + \ldots + a_p\mathbf{v}_p = \mathbf{b}$$

Q: What is the geometry of a span? What cases should be considered?

- The vector equation

$$a_1\mathbf{v}_1 + \ldots + a_p\mathbf{v}_p = \mathbf{b}$$

has the same solution as the linear system whose augmented matrix is

$$[\mathbf{v_1} \quad \mathbf{v_2} \quad \ldots \quad \mathbf{v_p} \quad \mathbf{b}]$$

In this case, the variables – the unknowns – would be the coefficients a_i, and a solution would consist of the appropriate possibilities of values for those coefficients.

Example: #9, p. 37

$$\begin{array}{rcrcrcl} & & x_2 & + & 5x_3 & = & 0 \\ 4x_1 & + & 6x_2 & - & x_3 & = & 0 \\ -x_1 & + & 3x_2 & - & 8x_3 & = & 0 \end{array}$$

Example: #12, p. 38. In this problem we throw you for another loop, by using the letter "a" for vectors! You have to pay attention, and not let us mess you up too badly just by poor notation....

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$$

- Two vectors are equal only if they have the same dimensions, and their components are the same.

- Algebraic properties of the **vector space** $\mathbb{R}^n$: for all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d,

  (i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

  (ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

  (iii) $\mathbf{u} + \mathbf{0} = \mathbf{u}$

  (iv) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(viii) $1\mathbf{u} = \mathbf{u}$

Example: #21, p. 38

Example: #27, p. 38