Section 1.5: Solution Sets of Linear Systems

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Abstract

This section shows us how to think of the solution set of a linear system geometrically, in terms of vectors. The main trick is to find the solution of a related system, the homogeneous system, and then find a particular solution to the system.

The solutions are some sorts of parametric representations of points (if only a trivial solution of the homogeneous equation exists), lines, planes, hyper-planes, etc.

The **homogeneous equation** $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution (that is, other than the zero vector $\mathbf{x} = \mathbf{0}$) if and only if the system of equations has at least one free variable.

Theorem 6: Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given vector $\mathbf{b}$, and let $\mathbf{p}$ be a particular solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Example: Proof (by linearity): #25, p. 56

(a) (Show that $\mathbf{w}$ is a solution.)

Suppose $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$, so that $A\mathbf{p} = \mathbf{b}$. Let $\mathbf{v}_h$ be any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$, and let $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$. Show that $\mathbf{w}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

$$A\underline{w} = A(\underline{p} + \underline{v}_h) = A\underline{p} + A\underline{v}_h$$
$$= \underline{b} + \underline{0} = \underline{b} \quad \checkmark$$

(b) (Show that $\mathbf{w}$ is the only type of solution.)

Let $\mathbf{w}$ be any solution of $A\mathbf{x} = \mathbf{b}$, and define $\mathbf{v}_h = \mathbf{w} - \mathbf{p}$. Show that $\mathbf{v}_h$ is a solution of $A\mathbf{x} = \mathbf{0}$. This shows that every solution of $A\mathbf{x} = \mathbf{b}$ has the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, with $\mathbf{p}$ a particular solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v}_h$ a solution of $A\mathbf{x} = \mathbf{0}$.

$$A(\underline{v}_h) = A(\underline{w} - \underline{p}) = A\underline{w} - A\underline{p}$$
$$= \underline{b} - \underline{b} = \underline{0}$$

So $\underline{v}_h$ is a soln of the homogeneous equation.

So $\underline{w} = \underline{p} + \underline{v}_h$ is the sum of a particular soln and a soln of the homogeneous equation.

Example: #8, p. 55 $\left[\begin{array}{cccc|c} 1 & -2 & -9 & 5 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{array}\right]$

$$\sim \begin{bmatrix} 1 & 0 & -5 & -7 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix}$$

$x_1 = 5x_3 + 7x_4$

$x_2 = -2x_3 + 6x_4$

$$\underset{2\times 4}{A}\ \underset{4\times 1}{\underline{x}} = \underset{2\times 1}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5x_3 + 7x_4 \\ -2x_3 + 6x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

$$A \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -9 & 5 \\ 0 & 1 & 2 & -6 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example: #9, p. 55 $\underset{2\times 3}{\begin{bmatrix} 3 & -9 & 6 \\ -1 & 3 & -2 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

$$\sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 3x_2 - 2x_3$

$$\underline{x} = \begin{pmatrix} 3x_2 - 2x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$A\underline{x} = \underline{b}$ – for what vector $\underline{b}$ will we have a consistent system? $\underline{b}$ is a linear combo of the column vectors, which are scalar multiples of each other; so $\underline{b}$ has to be a scalar multiple of $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

General soln: $\underline{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

Summary

You might relate the solutions of these equations to your history from calculus as follows:

$$[a_{11}a_{12}a_{13}] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [0]$$

is the same as

$$\langle a_{11}, a_{12}, a_{13} \rangle \cdot \langle x_1, x_2, x_3 \rangle = 0$$

It says that the row vector (which we might call $\mathbf{A}_1$) is perpendicular, or orthogonal, to the solution vector $\mathbf{x}$.

Then

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is the same as

$$\langle a_{11}, a_{12}, a_{13} \rangle \cdot \langle x_1, x_2, x_3 \rangle = 0$$

and

$$\langle a_{21}, a_{22}, a_{23} \rangle \cdot \langle x_1, x_2, x_3 \rangle = 0$$

i.e., that the $\mathbf{x}$ is orthogonal to both row vector ($\mathbf{A}_1$ and $\mathbf{A}_2$).

Now if

$$[a_{11}a_{12}a_{13}] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [b]$$

this says that

$$\langle a_{11}, a_{12}, a_{13} \rangle \cdot \langle x_1, x_2, x_3 \rangle = b.$$

That is, that the projection of $\mathbf{x}$ onto $\mathbf{A}_1$ is equal to b

You remember what this means: that

$$\mathbf{A}_1 \cdot \mathbf{x} = |\mathbf{A}_1||\mathbf{x}|\cos(\theta)$$

where θ is the angle between the vectors. Hence

$$A\mathbf{x} = \mathbf{b}$$

says: "the projections of $\mathbf{x}$ onto the rows of A make up the components of $\mathbf{b}$", and if

$$A\mathbf{x} = \mathbf{0}$$

then $\mathbf{x}$ is orthogonal to every row of A; or, alternatively

"$\mathbf{x}$ is orthogonal to the span of the row vectors of A".

The bang is still this: the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Example: #35, p. 56

$$\begin{bmatrix} -1 & -2 & 3 \\ -3 & -4 & 7 \\ -2 & -8 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \left(\text{or } \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

Example: #37, p. 56 – assumptions matter!

(consistency)

#26 Suppose $A\underline{x} = \underline{b}$ has a soln.
Explain why the soln is ! precisely when $A\underline{x} = \underline{0}$ has only the trivial soln.

Let $\underline{p}$ be a soln.

$\Rightarrow$ Suppose $\underline{p}$ is ! soln of $A\underline{x} = \underline{b}$.
Show that $A\underline{x} = \underline{0}$ has only the trivial soln.

By contradiction, assume $A\underline{x} = \underline{0}$ has a non-trivial soln., $\underline{v}_h \neq \underline{0}$. Then

$$A(\underline{p} + \underline{v}_h) = A\underline{p} + A\underline{v}_h = \underline{b} + \underline{0} = \underline{b}$$

is another soln of $A\underline{x} = \underline{b}$, which is a contradiction.

$\therefore$ $A\underline{x} = \underline{0}$ has only the trivial soln.

$\Leftarrow$ Assume that $A\underline{x} = \underline{0}$ has only the trivial solution, & that $A\underline{x} = \underline{b}$ is consistent. Then the soln of $A\underline{x} = \underline{b}$ is unique.

By contradiction: assume $A\underline{x} = \underline{b}$ has two distinct solns, $\underline{p}_1$ & $\underline{p}_2$. Then

$$A(\underline{p}_1 - \underline{p}_2) = A\underline{p}_1 + A\underline{p}_2 = \underline{b} - \underline{b} = \underline{0}$$

If $\underline{p}_1 \neq \underline{p}_2$, then $A\underline{x} = \underline{0}$ has a non-trivial soln, Contradiction. $\therefore$ $A\underline{x} = \underline{b}$ has a ! soln.