Section 1.7: Linear Independence

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Abstract

This section offers a different take on the equation $A\mathbf{x} = \mathbf{0}$ that we studied in section 1.5: we focus on the columns of A, and ask what relation must exist between them when a non-trivial solution of the homogeneous system exists.

If we think of the matrix A in terms of its column vectors (call them $\mathbf{a}_i$), then

$$A\mathbf{x} = \mathbf{0} \iff x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0} \tag{1}$$

Definition: a set of vectors $S = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is **linearly independent** if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (that is, if component $x_i = 0 \ \ \forall i$). Otherwise the set is **linearly dependent**.

In matrix terms: The columns of A are linearly independent $\iff$ $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Example: #3, p. 71 $\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \end{bmatrix}$

$$\begin{bmatrix} -3 \\ 9 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$3\begin{bmatrix} 1 \\ -3 \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \end{bmatrix} = \underline{0}$$

Example: #4, p. 71 $\begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -8 \end{bmatrix}$

$$\begin{bmatrix} -1 \\ 4 \end{bmatrix} \neq \alpha\begin{bmatrix} -2 \\ -8 \end{bmatrix} \Rightarrow \text{independence}$$

$$\begin{bmatrix} -1 & -2 & 0 \\ 4 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 & 0 \\ 0 & -16 & 0 \end{bmatrix}$$

$\Rightarrow$ only the trivial soln.

Example: #1, p. 71 $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$

$$\begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -6 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

no free variables $\Rightarrow$

independence

Theorem 7: An indexed set $S = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ of two or more vectors is linearly **dependent** $\iff$ at least one of the vectors in S is a linear combination of the others.

The upshot of Theorem 7 is that one of the vectors of a dependent set S can be expressed as a linear combination of the others. From (1) we deduce that one of the coefficient x_i is non-zero: if $x_i \neq 0$, then we can solve for $\mathbf{a}_i$ as

$$\mathbf{a}_i = \frac{-1}{x_i}\left(x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_{i-1}\mathbf{a}_{i-1} + x_{i+1}\mathbf{a}_{i+1} + \ldots + x_n\mathbf{a}_n\right)$$

If S contains the zero vector, then the set is linearly dependent, since we can set the coefficient of the zero vector to 1, set all the other coefficients to zero, and we have a non-trivial solution to the homogeneous equation.

Example: #17, p. 71 $\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 4 \end{bmatrix}$

$$0\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} -6 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & -6 \\ 5 & 0 & 5 \\ -1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It should also be clear from our discussions of spans that if we have more vectors in the set than the dimension of the space in which the vectors reside then the set will be linearly dependent. For example, if you have four distinct vectors in three-space, then the set will be linearly dependent. This is the substance of Theorem 8:

Theorem 8: $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Example: #8, p. 71 $\begin{bmatrix} 1 & -3 & 3 & -2 \\ -3 & 7 & -1 & 2 \\ 0 & 1 & -4 & 3 \end{bmatrix}$

4 vectors in 3-space.

That's more vectors than I need to generate R^3 (so at least one's extraneous.

Example: #11, p. 71 $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ h \end{bmatrix}$

Think "non-trivial soln of $A\underline{x} = \underline{0}$" (free variable)

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ -1 & -5 & 5 & 0 \\ 4 & 7 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & -5 & 4+h & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & -6+h & 0 \end{bmatrix}$$

$h = 6 \Rightarrow$ free variable

$\Rightarrow$ linear dependence.

Example: #41, p. 72