Section 2.2: The Inverse of a Matrix

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Abstract

The inverse of a matrix is analogous to the multiplicative reciprical: we want to solve $A\mathbf{x} = \mathbf{b}$, and so we'd like to say that $\mathbf{x} = \mathbf{b}/A$ - but we don't know how to say that with matrices! Let's find out....

First of all, this concept only applies when matrices are **square**: so only n x n matrices could possibly be invertible.

Definition: An n x n matrix A is **invertible** if there exists an n x n matrix C (the **inverse** of A) such that

$$CA = I = AC$$

$$\left.\begin{aligned} C_{m\times n} A_{n\times m} &= I_{m\times m} \\ A_{n\times m} \cdot C_{m\times n} &= I_{n\times n} \end{aligned}\right\} \Rightarrow m = n \text{ square}$$

The inverse C is denoted A^{-1}, and is unique. A square matrix for which the inverse fails to exist is called **singular**.

A simple formula exists for the inverse of a two-by-two matrix: if A is given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then, provided $ad - bc \neq 0$,

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Otherwise, if $ad - bc = 0$, then A is singular. The quantity $ad - bc$ is called the **determinant** of A: $det(A) \equiv ad - bc$.

Example: #1, p. 126 (check!)

$$A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix} \qquad det(A) = 2 = 8\cdot 4 - 6\cdot 5 \neq 0$$

$\Rightarrow$ A^{-1} exists,

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix}$$

$$A \cdot A^{-1} = ? = \frac{1}{2} \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem: 5 if A is invertible, then $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$: $\mathbf{x} = A^{-1}\mathbf{b}$.

Example: #5, p. 126 (check!)

$$\begin{aligned} 8x_1 + 6x_2 &= 2 \\ 5x_1 + 4x_2 &= -1 \end{aligned} \longrightarrow \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$A^{-1}A\cdot \underline{x} = \frac{1}{2}\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\underline{x} = \underline{I}\cdot \underline{x} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}$$

Theorem: 6

(a) If A is invertible, then $(A^{-1})^{-1} = A$. ("A is the inverse of A^{-1}.")

Example: #1, p. 126 (check!)

$$A^{-1} = \begin{bmatrix} 4/2 & -6/2 \\ -5/2 & 8/2 \end{bmatrix} \qquad \det(A^{-1}) = \frac{4}{2}\cdot\frac{8}{2} - \left(\frac{-6}{2}\right)\left(\frac{-5}{2}\right)$$

$$= \frac{1}{4}\cdot 2 = \frac{1}{2}$$

$$(A^{-1})^{-1} = 2\begin{bmatrix} 8/2 & 6/2 \\ 5/2 & 4/2 \end{bmatrix} \qquad \left(\text{Note: } \det(A^{-1}) = \frac{1}{\det(A)}\right)$$

$$= \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix} = A.$$

(b) If A and B are n x n invertible matrices, then so is AB, and the inverse of AB is the product of the inverses, in the reverse order:

$$(AB)^{-1} = B^{-1}A^{-1}$$

More generally, the inverse of a product of any number of invertible matrices is the product of the inverses in reverse order.

$$(AB)^{-1}\cdot AB = I$$

$$B^{-1}\underbrace{A^{-1}\cdot A}_{I}\cdot B =$$

$$B^{-1}I\cdot B =$$

$$B^{-1}\cdot B = I.$$

Example: #15, p. 126

ABC, $A, B,$ + C invertible $n\times n$ matrices.

Claim: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

$$\underbrace{C^{-1}\underbrace{B^{-1}\underbrace{A^{-1}A}_{I}B}_{I}C}_{I}.$$

$ABC\underline{x}$

(c) If A is invertible, then so is A^T, and the inverse of A^T is the transpose of A^{-1}:

$$(A^T)^{-1} = (A^{-1})^T$$

The inverse of the transpose is the transpose of the inverse.

Definition: an **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix. Each elementary matrix is invertible.

If an elementary row operation is performed on an m x n matrix A, the resulting matrix can be written as EA, where the m x m matrix E is created by performing the same row operation on I_m.

Example: #28, p. 127

$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \end{array} \sim \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 0 & -3 \end{bmatrix} \begin{array}{l} 1 \\ 2 \\ 3 - 4 \cdot 1 \end{array}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \qquad EA = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 0 & -3 \end{bmatrix}$$

Theorem: 7 Matrix $A_{n \text{ x } n}$ is invertible if and only if A is row equivalent to I_n. The elementary row operations that transform A into I_n simultaneously transforms I_n into A^{-1}.

Theorem 7 suggests a method for finding A^{-1}: row reduce the augmented matrix $[AI_n]$. If A is row equivalent to I_n, then $[AI_n]$ is row equivalent to $[I_nA^{-1}]$.

Example: #1, p. 126

$$\left[\overbrace{\begin{matrix} 8 & 6 \\ 5 & 4 \end{matrix}}^{A} \;\; \overbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}^{I_2}\right] \sim \begin{bmatrix} 8 & 6 & 1 & 0 \\ 0 & \frac{2}{8} & -\frac{5}{8} & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 8 & 6 & 1 & 0 \\ 0 & 2 & -5 & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 8 & 0 & 16 & -24 \\ 0 & 2 & -5 & 8 \end{bmatrix} \sim \left[\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2} \;\; \underbrace{\begin{matrix} 2 & -3 \\ -\frac{5}{2} & 4 \end{matrix}}_{A^{-1}}\right]$$

Claim:

$$A^{-1} = \begin{bmatrix} \frac{4}{2} & -\frac{6}{2} \\ -\frac{5}{2} & \frac{8}{2} \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -\frac{5}{2} & 4 \end{bmatrix}$$

Example: #18, p. 126

P invertible and $A = PBP^{-1}$,

Solve for B in terms of A.

$$P \cdot P^{-1} = I = P^{-1}P$$

$$AP = PBP^{-1}P = PBI = PB$$

$$P^{-1}AP = P^{-1}(PB) = P^{-1}PB = IB = B$$

$$\boxed{B = P^{-1}AP}$$

Example: #19, p. 126

$C^{-1}(A+X)B^{-1} = I$ has a soln, X:

$$C \cdot C^{-1}(A+X)B^{-1} = C \cdot I$$

$$I \cdot (A+X)B^{-1} = C$$

$$(A+X)B^{-1} = C$$

$$(A+X)B^{-1} \cdot B = CB$$

$$(A+X) = CB$$

$$\boxed{X = CB - A}$$

Example: #21, p. 126 Why are the columns of A linearly independent when A is invertible?

A is row-equivalent to I_n, with n basic variables (n pivots); so A must have n basic variable (n pivots). So it can't have any linearly dependent columns.

Note : A^{-1} is generally not calculated: we don't need to know its entries to solve $A\mathbf{x} = \mathbf{b}$ (similar to the notion that we don't need to row reduce to reduced row echelon form to solve: we can stop with a triangular matrix).