Chapter 3 Summary: The Determinant

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Abstract

As a preface, note that we're always talking about square matrices in the following.

This is certainly the most important fact: the determinant's size represents the volume of the image of the ball of unit volume under the linear transformation represented by $A\mathbf{x}$. So if the determinant is zero, the ball's image has been "squashed" so that it has zero volume. This means that the matrix is singular, and cannot be inverted. This is the key fact, and the fact that we will encounter again when we bump up against **eigenvalues**.

We first encountered the determinant when inverting 2 x 2 matrices: it appears in the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The determinant is denoted $\det A$, and $\det A = ad - bc$. Obviously if $\det A = 0$, then the 2 x 2 matrix is not invertible, so the determinant was mixed up with the idea of invertibility.

Theorem 2, p. 189: If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

Upper T :

Lower T :

Diagonal :

Theorem 3, p. 192: Let A be a square matrix.

(a) If a multiple of one row of A is added to a different row to produce a matrix B, then det $B =$ det A.

(b) If two rows of A are interchanged to produce B, then det $B =$ -det A.

(c) If one row of A multiplied by k to produce B, then det $B =$ $k\cdot$ det A.

These are the operations of elementary matrices, so we see how to relate the determinant of the matrix U obtained by row-reduction from A to the determinant of A itself:

Example: Let's check Theorem 3 against 2 by 2 matrices. Suppose that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

① Row sum: $A \sim \begin{bmatrix} a & b \\ c+ae & d+be \end{bmatrix} = B$ $\quad \det B = a(d+be) - b(c+ae) = ad - bc + abe - abe = \det A$

② interchange: $A \sim \begin{bmatrix} c & d \\ a & b \end{bmatrix} = B$

$\det B = cb - ad = -\det A$

③ Scalar multiple of a row:

$A \sim \begin{bmatrix} ka & kb \\ c & d \end{bmatrix} = B \quad \det B = kad - kbc = k \det A$

Formula (1), p. 194: Suppose a square matrix A has been reduced to an echelon form U by row replacements and r row interchanges. Then

$$\det A = (-1)^r \underbrace{\det U}_{\text{easy - upper triangular}}$$

Since U is triangular, its determinant is simply the product of its diagonal entries.

$$E_p \cdots E_1 A = U \implies A = (E_p \cdots E_1)^{-1} U = \underbrace{E_1^{-1} \cdots E_p^{-1}}_{L} U$$

What do the inverses of the elementary matrices look like?

E_i interchange; $\quad E_i \cdot E_i = I \quad E_i^{-1} = E_i$!

E_j replacement:

$$E_j = \left[\begin{array}{c|c|c} \ddots & 0 & 0 \\ \hline 0 & \begin{smallmatrix} 1 & & \\ & \ddots & \\ k & & 1 \end{smallmatrix} & 0 \\ \hline 0 & 0 & \ddots \end{array}\right] \qquad E_j^{-1} = \left[\begin{array}{c|c|c} \ddots & 0 & 0 \\ \hline 0 & \begin{smallmatrix} 1 & & \\ & \ddots & \\ -k & & 1 \end{smallmatrix} & 0 \\ \hline 0 & 0 & \ddots \end{array}\right]$$

$B = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}$

Theorem 4, p. 194: $A_{n \text{ x } n}$ invertible $\iff$ det $A \neq 0$.

Theorem 5, p. 196: If A is an n x n matrix, then det $A^T =$ det A.

(Check 2 x 2 matrices.) Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\begin{aligned} \det(A^T) &= ad - cb \\ &= ad - bc \\ &= \det A \end{aligned}$$

Theorem 6, p. 196:

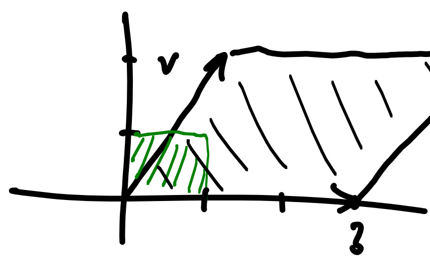
$$A_{n \text{ x } n}, B_{n \text{ x } n} \Longrightarrow \det AB = (\det A)(\det B)$$

Properties/Tricks/Hints/Etc.

In $\mathbb{R}^n$ the ball of unit volume (that is, a ball centered at the origin of volume 1) is transformed under a linear transformation into an ellipsoid. The volume of the ellipsoid is the absolute value of the determinant. If you like, you can consider that the **definition** of the determinant! See the figure on page 209.

The determinant was important classically, perhaps more so than it is today. The idea of cofactors is classical, elegant, but not particularly practical. As mentioned in the text, the calculation of a determinant is carried out by the method of LU decomposition, and relies upon the simple fact that the determinant of a triangular (and especially diagonal) matrix is the product of the diagonal elements.

#41 p 191 $\quad u = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$



Area = 6

$\det(A) = 6$

$$\left.\begin{array}{l} \hat{\imath} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array}\right\} \text{These generate the square}$$

$$A\begin{bmatrix} \hat{\imath} & \hat{\jmath} \end{bmatrix} = A\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A$$

$$A\hat{\imath} = u \qquad A\hat{\jmath} = v$$

A square of unit volume (i.e. area) gets transformed into a parallelogram of volume (i.e. area) 6.