Section 4.2: Null spaces, column spaces, and linear transformations

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Abstract

We examine various subspaces which are naturally defined by a matrix A.

The solution set of the homogeneous equation $A_{m \text{ x } n}\mathbf{x} = \mathbf{0}$ forms a subspace of $\mathbb{R}^n$, as one can easily see:

(a) the zero vector is in the solution set (the trivial solution);

(b) Consider two vectors in the solution set, $\mathbf{u}$ and $\mathbf{v}$: then $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u}+A\mathbf{v} = \mathbf{0}+\mathbf{0} = \mathbf{0}$, so the solution set is closed under addition.

(c) Consider a vectors in the solution set, $\mathbf{u}$ and an arbitrary constant c: then $A(c\mathbf{u}) = cA\mathbf{u} = \mathbf{0}$, so the solution set is closed under scalar multiplication.

Definition: Null space of an m x n matrix A: the null space of an m x n matrix A, denoted Nul A, is the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. It is the set of all $\mathbf{x} \in \mathbb{R}^n$ that are mapped to the zero vector of $\mathbb{R}^m$ by the transformation $\mathbf{x} \longrightarrow A\mathbf{x}$.

Theorem 2: : The null space of an m x n matrix A is a subspace of $\mathbb{R}^n$.

Example: #3, p. 234.

$$\underset{2\times4}{A} = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix} \qquad A\underline{x} = \underline{0}$$

$$\begin{bmatrix} 1 & 3 & 5 & 0 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 & 6 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{bmatrix}$$

$x_1 + -7x_3 + 6x_4 = 0$ $\qquad$ $x_1 = 7x_3 - 6x_4$

$x_2 + 4x_3 - 2x_4 = 0$ $\qquad$ $x_2 = -4x_3 + 2x_4$

x_3, x_4 free

$$x_3 \overset{\underline{v}_1}{\begin{pmatrix} 7 \\ -4 \\ 1 \\ 0 \end{pmatrix}} + x_4 \overset{\underline{v}_2}{\begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}}$$

This is the Null space

the span of $\underline{v}_1 + \underline{v}_2$

Notice that the number of vectors in the spanning set for Nul A equals the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.

Definition: Column space of A **:** Another subspace associated with the matrix A is the column space, Col A, defined as the span of the columns of A: Col A = Span $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. As a span, it is clearly a subspace (Theorem 3).

Col $A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x}$ for some $\mathbf{x}$ in $\mathbb{R}^n\}$, which says that Col A is the range of the transformation $\mathbf{x} \longrightarrow A\mathbf{x}$.

Definition: Row space of A **:** The column space of A^T

$\mathbb{R}^n$, but perpendicular to

The null space of A lives in the row space of A. ← important!

Example: #16, p. 234

$$\left\{ \begin{bmatrix} b-c \\ 2b+c+d \\ 5c-4d \\ d \end{bmatrix} : b, c, d \in \mathbb{R} \right\}$$

$$\text{Col } A = b\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix},$$

the column space of

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

(A is not unique - you could permute columns)

Example: #22, p. 235

$\begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}$ works as an element of Nul A.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in$ Col A

Definition: Linear Transformation : A linear transformation T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

(a) $\mathrm{T}(\mathbf{u}+\mathbf{v}) = \mathrm{T}(\mathbf{u}) + \mathrm{T}(\mathbf{v})$

(b) $\mathrm{T}(\mathrm{c}\mathbf{u}) = \mathrm{cT}(\mathbf{u})$

The **kernel** (or **null space**) of T is the set of $\mathbf{u}$ such that $T(\mathbf{u}) = \mathbf{0}$. The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V. Effectively the column space — Span(V)

Example: #30, p. 235

Prove that the range of T is a subspace of W. Let $T(\underline{x})$ and $T(\underline{w})$ be images of vectors $\underline{x}$ + $\underline{w}$ of V.

$T(\underline{0}) = \underline{0}$ for linear transformations, so

$\underline{0} \in$ range.

Is $T(\underline{x}) + T(\underline{w})$ in the range?

$T(\underline{x}) + T(\underline{w}) = T(\underline{x}+\underline{w})$, so the sum's in the range.

Is $cT(\underline{x})$ in the range? Yes: $T(c\underline{x}) = cT(\underline{x})$ ✓

Examples of linear transformations include matrix transformations, as well as differentiation in the vector space of differentiable functions defined on an interval (a, b).

Example: Example #8, p. 233

V - vector space of all real-valued functions f on [a, b]

$D: V \to W$,

where W is the space of continuous functions on [a, b]

(V is a subspace of W - you've got to be continuous to be differentiable).

$D(\underline{0}) = \underline{0}$

0 vector, 0 function

$$D(a\cdot \underline{f} + b\cdot \underline{g}) = a\cdot D(\underline{f}) + b\, D(\underline{g}).$$

V + W are infinite dimensional!

And there's a kernel - constant functions have zero derivatives: they get annihilated.

#34 p 234

$$T: C[0,1] \to C[0,1]$$

Given $\underline{f} \in C[0,1]$; $T(\underline{f}) = \underline{F} = \int \underline{f}(t)\,dt$ / $\underline{F}(0) = 0$.

Show that T is linear:

$$T(\underbrace{\alpha \underline{f} + \beta \underline{g}}_{\in C[0,1]}) = \alpha T(\underline{f}) + \beta T(\underline{g})$$

$$T(\alpha \underline{f} + \beta \underline{g}) = \int [\alpha f(t) + \beta g(t)]\,dt \quad / \left(\begin{smallmatrix} =0 \\ \text{at } 0\end{smallmatrix}\right)$$

$$= \alpha \int \underline{f}(t)\,dt + \beta \int \underline{g}(t)\,dt$$

$$= \alpha \underline{F}(t) + \beta \underline{G}(t)$$

$$= \alpha T(\underline{f}) + \beta T(\underline{g})$$

✓

Note: $\alpha F(0) + \beta G(0) = 0$