Section 4.3: Linearly Independent Sets; Bases

March 5, 2008

Abstract

We're accustomed to writing vectors in terms of a set of fixed vectors: for example, in two-space we write every vector in terms of vectors **i** and **j**. Each vector has a unique representation in terms of these two vectors, which is important. This set of two vectors is called a **basis** of two-space: it is enough vectors to write each vector of the space in terms of it, but not so many vectors that there are multiple representations of each vector. These are the two important properties: spanning the space, and avoiding any redundency. That is, a basis is the smallest spanning set possible. It is also the largest set of linearly independent vectors: any more, and you'd have dependence.

Definition: linear independence : (Recall: independence if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.) Alternatively, an indexed set of vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ (columns of matrix A) is said to be linearly independent if the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0} \tag{1}$$

has *only* the trivial solution ($x_1 = x_2 = \ldots = x_n = 0$). The set is **linearly dependent** if equation (1) has a nontrivial solution.

Example: #4, p. 243 (check for independence)

$$A = \begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -7 & 0 \\ -2 & -3 & 5 & 0 \\ 1 & 2 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -7 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 3/2 & 15/2 & 0 \end{bmatrix}$$

$\therefore$ independence – 3 pivot positions

Theorem 4: An indexed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of two or more vectors, $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent $\iff$ some $\mathbf{v}_j$ $(j > 1)$ is a linear combination of the preceding vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}\}$.

Example: #33, p. 245 $p_1(t) = 1 + t^2$ and $p_2(t) = 1 - t^2$. Is $\{p_1, p_2\}$ a linearly independent set in P_3?

Is p_2 a scalar multiple of p_1?

Is $1 - t^2 = \alpha(1 + t^2)$

If so $1 = \alpha \Rightarrow \alpha = 1$

and $-t^2 = \alpha t^2 \Rightarrow \alpha = -1$ No

$\therefore$ $\{p_1, p_2\}$ is an independent set in P_3.

Definition: Basis Let H be a subspace of a vector space V. An indexed set of vectors $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a basis for H if

(a) B is a linearly independent set, and

(b) the subspace spanned by B coincides with H; that is,

$$H = \text{Span } \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$$

In the vector space of all polynomials, P, $P_3 = \text{Span}\{1, t, t^2, t^3\}$ & they're independent → hence a basis.

Example: #4, p. 243 (check if it's a basis)

Example: #34, p. 245 $p_1(t) = 1 + t$, $p_2(t) = 1 - t$, and $p_3(t) = 2$. Dependent? Find a basis for $Span\{p_1, p_2, p_3\}$.

Example: The columns of the n x n identity matrix I_n for a basis, called the **standard basis** for $\mathbb{R}^n$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

In three-space these are simply the vectors **i**, **j**, and **k**.

Let $\underline{v} \in \mathbb{R}^n$: $\underline{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = v_1 \cdot \underline{e}_1 + v_2 \cdot \underline{e}_2 + \ldots + v_n \underline{e}_n$

Theorem 5 (the spanning set theorem): Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V, and let $H = \text{Span}\ \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

(a) If one of the vectors in S – say $\mathbf{v}_k$ – is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

(b) If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

Theorem 6: the pivot columns of a matrix A form a basis for Col A.

Turns out that elementary row operations on a matrix do not affect the linear dependence relations among the columns of the matrix. Hence, the reduced matrix has the same independent columns as the original matrix. **Warning**: Make sure to choose the columns of the matrix A, however, rather than the columns of the reduced matrix!

Example: #36, p. 245

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 2 \\ 2 \\ 7 \\ -3 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 8 \\ -4 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ -2 \\ 9 \\ -5 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 4 \\ 6 \\ -2 \end{bmatrix}$$

$\{u_1, u_2\}$ is a basis for H | rref(U)

$\{v_1, v_2\}$ " " " " K | rref(V)

$\{u_1, u_2, v_1\}$ " " " " $H+K$ | rref([U V])

(do the row reduction!)