Section 4.4: Coordinate Systems

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Abstract

A basis gives us a way of writing each vector **v** in a vector space in a unique way, as a linear combination of the basis vectors. The coefficients of the basis vectors can be considered the coordinates of **v** in a coordinate system determined by the basis vectors.

Theorem 7: the Unique Representation Theorem

Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$$

Suppose not: $= \gamma_1 \underline{b}_1 + \ldots + \gamma_n \underline{b}_n$ Suppose $c_1 \neq \gamma_1$

$(c_1 - \gamma_1)\underline{b}_1 = -\left[(c_2-\gamma_2)\underline{b}_2 + \ldots + (c_n - \gamma_n)\underline{b}_n\right] \Rightarrow$ dependence!

Don't have dependence, contradiction.

Definition: Coordinates Suppose $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V, and $\mathbf{x}$ is in V. The **coordinates of x relative to the basis** B are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$.

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the **coordinate vector of** x **(relative to** B), or the B**-coordinate vector of** x. The mapping

$$\mathbf{x} \mapsto [\mathbf{x}]_B$$

is the **coordinate mapping (determined by** B).

Example: #1, p. 253

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}, \quad [\underline{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

The invisible basis

$$\underline{x} = 5\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 3\begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}.$$

$\underline{x}$ looks completely different in the two bases (one of which is unspecified)

Let

$$P_B = [\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_n]$$

$$\left(\begin{array}{l} \text{e.g.} \\ P_B = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix} \\ \underline{x} = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix} \end{array} \right.$$

Then

$$\mathbf{x} = P_B[\mathbf{x}]_B$$

is the link between the standard basis representation of $\mathbf{x}$ (on the left) and the representation of $\mathbf{x}$ in the basis B.

This suggests that

$$P_B^{-1}\mathbf{x} = [\mathbf{x}]_B$$

is the link in the opposite direction (and it is!).

Example: #5, p. 254

$$\underline{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad \underline{b}_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix} \quad \underline{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix} (x)_B$$

Solve this linear system: $\begin{bmatrix} 1 & 2 & -2 \\ -3 & -5 & 1 \end{bmatrix} \sim$

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -5 \end{bmatrix} \qquad \begin{array}{l} x_1 = -2 - 2(x_2) = 8 \\ x_2 = -5 \end{array}$$

$$(x)_B = \begin{bmatrix} 8 \\ -5 \end{bmatrix}_B$$

Example: #14, p. 254

$B = \{1 - t^2, t - t^2, 2 - 2t + t^2\}$ is a basis for P_2. Find $[p(t)]_B$, where

$$p(t) = 3 + t - 6t^2.$$

$$= c_1 \underline{b}_1 + c_2 \underline{b}_2 + c_3 \underline{b}_3$$

$$\underline{b}_1 \equiv 1 - t^2 \qquad \underline{b}_2 \equiv t - t^2 \qquad \underline{b}_3 \equiv 2 - 2t + t^2$$

Standard basis: $\{1, t, t^2\}$ $\qquad \underline{p} = \begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix}$

$$\overbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix}}^{P_B} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix}$$

solve for the coefficients c_i

Theorem 8: Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space

V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_B$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

This is an example of an *isomorphism* ("same form") from V onto W. These spaces are essentially indistinguishable.

Example: #23, p. 254

Example: #24, p. 254