Section 4.5: The Dimension of a Vector Space

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Abstract

The dimension of 2-space is 2; the dimension of 3-space is 3! Dimension is really pretty obvious in a lot of ways: how many "degrees of freedom" do you have to move around in? The dimension of $\mathbb{R}^n$ is n; so we understand finite dimensional spaces pretty well. What's the dimension of the vector space of all polynomials, P? Yes, Virginia, we can have infinite-dimensional spaces....

Theorem 9: If a vector space V has a basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Consider $\{\underline{v}_1, \ldots, \underline{v}_p\}$ $p > n$.

$$\begin{bmatrix} c_{11} & c_{21} & \cdots & c_{p1} \\ c_{12} & c_{22} & \cdots & c_{p2} \\ c_{13} & c_{23} & \cdots & c_{p3} \\ \vdots & \vdots & & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow$$

dependence ($p > n$ ⇒ free variables ⇒ non-trivial null space)

Theorem 10: : If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

$n-1$ vectors in a basis, then by Thm 9 n vectors would be dependent - so we'd have no n-vector bases.

Definition: dimension of a vector space If V is spanned by a finite set, then V is **finite-dimensional**, and the dimension V (dim V) is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

Example: #2, p. 260

$$\left\{ \begin{bmatrix} 4s \\ -3s \\ -t \end{bmatrix} : s, t \in \mathbb{R} \right\} \qquad \begin{bmatrix} 4s \\ -3s \\ -t \end{bmatrix} = s \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Basis: $\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$ (Not unique - obvious perhaps....)

Theorem 11: Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

Example: **#11, p. 261**

$$\underbrace{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}3\\1\\1\end{bmatrix}}_{\text{clearly linearly independent}}, \begin{bmatrix}9\\4\\-2\end{bmatrix}, \begin{bmatrix}-7\\-3\\1\end{bmatrix}$$

know: dependence

$$\begin{bmatrix}1 & 3 & 9 & -7\\0 & 1 & 4 & -3\\2 & 1 & -2 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 9 & -7\\0 & 1 & 4 & -3\\0 & -5 & -20 & 15\end{bmatrix}$$

$\Rightarrow$ 2 dimensional (3rd row scalar multiple of 2nd)

Choose basis from pivot positions.

Theorem 12 (the Basis Theorem): Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

Example: **#22, p. 261**

$L_1 = 1$

$L_2 = 1 - t$

$L_3 = 2 - 4t + t^2$

$L_4 = 6 - 17t + 9t^2 - t^3$

$e_1 = 1$

$e_2 = t$

$e_3 = t^2$

$e_4 = t^3$

$$\begin{bmatrix}L_1\\L_2\\L_3\\L_4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\1 & -1 & 0 & 0\\2 & -4 & 1 & 0\\6 & -17 & 9 & -1\end{bmatrix}\begin{bmatrix}e_1\\e_2\\e_3\\e_4\end{bmatrix} \Rightarrow \underline{e} = A^{-1}\underline{L}$$

A - invertible, $\det A \neq 0$, linearly independent columns, ...

Let A be an m x n matrix. Then the dimension of Nul A is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$, and the dimension of Col A is the number of pivot columns in A.

Example: #14, p. 261

$$A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim \text{Nul } A = 3$

$\dim \text{Col } A = 3$

Example: #27, 28, p. 262

#28 Show that $C(\mathbb{R})$ (continuous functions on $\mathbb{R}$) is infinite dimensional.

$\mathbb{P}$ is a subspace of $C(\mathbb{R})$, & it's infinite dimensional.