Section 4.6: Rank

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Definition: Rank The rank of a matrix is the dimension of the column space of A. That is, it is equal to the number of independent vectors among the columns of the matrix.

Definition: row space the row space of a matrix A is the span of the rows of A.

Theorem 13: If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the non-zero rows of B form a basis for the row spaces of A and B.

Theorem 14 (The Rank Theorem): The dimensions of the column space and the row space of an m x n matrix A are equal (the rank of A). The rank satisfies the relation

$$\text{rank } A + \dim \text{Nul } A = n$$

$$\underbrace{\begin{bmatrix} & A & \end{bmatrix}}_{m \times n} \begin{bmatrix} \\ \underline{x} \\ \\ \end{bmatrix} = \underline{0}_m$$

Null space vectors live in $\mathbb{R}^n$; $\perp$ the Row A.

You may be wondering why the Nul space popped up here: all these spaces are **fundamentally connected**. The Nul A and the Row A make up all of $\mathbb{R}^n$.

Example: #2, p. 269

$$A \sim \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank A: 3

dim Nul A: 2

$$\mathcal{B}_{\text{Col } A} = \left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -4 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix} \right\};$$

$\mathcal{B}_{\text{Nul } A}$ – find solutions to

$$A\underset{\substack{\uparrow \\ \in \mathbb{R}^5}}{\underline{x}} = \underset{\substack{\uparrow \\ \in \mathbb{R}^4}}{\underline{0}}$$

$$\mathcal{B}_{\text{Row } A} = \left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} \right\}$$

Theorem : Invertible Matrix Theorem (continued): Let A be an n x n matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix:

- The columns of A form a basis of $\mathbb{R}^n$.
- Col $A = \mathbb{R}^n$
- dim Col A = n
- rank A = n
- Nul $A = \{\mathbf{0}\}$
- dim Nul $A = 0$

Example: #5,8-11, p. 269

#8 $\quad 5 \times 6$, 4 pivots ; dim Nul A = ? 2

Col A = $\mathbb{R}^4$? No

(columns are in $\mathbb{R}^5$)

#9 $\quad 5 \times 6$, dim Nul A = 4 ; dim Col A = ? 2

#10 $\quad 7 \times 6$, dim Nul A = 5 $\quad$ dim Col A = ? $\underline{1}$

#11 $\quad 8 \times 5$, dim Nul A = 2 $\quad$ dim Row A = 3

Example: #16, p. 269

$A_{6 \times 4}$ - whats minimum possible

dim Nul A ?

0 :

(greatest possible rank : 4)

rank + dim Nul = 4

Example: #18, p. 270

a. F - pull the corresponding columns of A

b. F - don't use the same strategy for finding a basis for the row space.

Example: #24, p. 270

$A_{7\times 6}$; Consider $A\underline{x} = \underline{b}$

Possible to have ! soln for some RHS?

Yes - A may have 6 pivots $\Rightarrow$ columns are independent $\Rightarrow$ $A\underline{x} = \underline{0}$ has only trivial soln. $\Rightarrow$ If $\exists$ solution, it's unique.

6 vectors cannot a 7-dimensional world make. Not possible for every RHS to have a soln.

#35 a.

$A_{5\times 7}$

a. C - columns are basis for Col A

Strategy:

1. rref (A)
2. find pivot columns
3. grab corresponding columns from A,
4. glue them together to make C.

$C_{5 \times \text{rank}(A)}$

$N_{7\times(7-\text{rank}(A))}$

1. rref (A)
2. Looking at 7-rank (A) free variables
3. Solve $A\underline{x} = \underline{0}$
4. The solns are a basis for Nul (A)
5. Glue them together lengthwise to make N.

R

① rref (A)

② Pick off the non-zero rows

③ Glue the rows together to make R

$R_{\text{rank}(A) \times 7}$

b. Consider $(A^T)_{7\times5}$

$$S = [R^T N]_{7\times7} = [R^T_{7\times \text{rank}(A)} \quad N_{7\times(7-\text{rank}(A))}]$$

$$T = [C \; M] = [C_{5\times\text{rank}(A)} \quad M_{5\times(5-\text{rank}(M))}]$$

M – Null Space of A^T

$A^T \underline{x} = \underline{0} \qquad A^T_{7\times5}\, \underline{x} = \underline{0}$

(rank A^T = rank A)

$M_{5\times(5-\text{rank}(A))}$

$T_{5\times5}$

$$\underset{\sim}{AN} = O_{5\times(7-\text{rank}(A))}$$

columns are a basis for everything that gets crushed.

What doesn't get crushed?

row space of A

R^T – columns form a basis for that which doesn't get crushed

Together the $(7-\text{rank}A) + \text{rank}A = 7$ vectors form a basis for $\mathbb{R}^7$, so S is invertible.