Question for today:
What's a good way to find the eigenvalues of a matrix?

Answer for today:
We use what is called the "characteristic equation."

1. Illustrative Example

Let $A = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$. What are the eigenvalues of A? Well, λ is an eigenvalue precisely when

$$(A - \lambda I)\underline{x} = 0 \qquad A\underline{x} = \lambda \underline{x}$$

has a nontrivial solution. Therefore, we can further analyze this equation to find eigenvalues.

Consider the following question: When does the above equation have nontrivial solutions? Well, $(A - \lambda I)$ is a matrix. If $(A - \lambda I)\underline{x} = 0$ has nontrivial solutions, what do we know about $(A - \lambda I)$? We know that it is *not* invertible.

Let's continue with this line of reasoning. If a matrix is not invertible, what else do we know? We know that its determinant is zero. That is,

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{bmatrix}\right) = (4-\lambda)(1-\lambda) + 2$$

$$= \lambda^2 - 5\lambda + 6$$

$$= (\lambda - 3)(\lambda - 2)$$

So what does this tell us? This tells us that the determinant is zero precisely when $\lambda = 3$ or $\lambda = 2$. That is, the matrix $(A - \lambda I)$ is not invertible for precisely these two values of λ, which tells us that these are precisely the eigenvalues of A.

This was only a 2×2 example, but let's go ahead and take a look at the general method:

Theorem Given an $n \times n$ matrix A, we find the eigenvalues of A by finding the numbers λ that satisfy the *characteristic equation* for A. The characteristic equation for A is given by

$$\det(A - \lambda I) = 0$$

$\Rightarrow$ n^{th} degree polynomial on the left.

This is nice, but in order to put it into practice we need to know how to calculate determinants. Recall the following:

Given a matrix A, we can perform a total of r row replacement and row interchange operations to obtain U, an echelon form of A. We do *not* allow scalings in this process, so the pivots of U are probably not equal to 1. Let u_{kk} denote the diagonal entries of U. The determinant of A is then defined as

$$\det A = (-1)^r \cdot (u_{11}u_{22} \dots u_{nn})$$

Note that if even one diagonal entry of U is zero, then $\det A = 0$. Let's do an example:

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & -4 \\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 7 & -4 \\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 7 & -4 \\ 0 & 0 & 3+\frac{8}{7} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 0 \\ 0 & 7 & -4 \\ 0 & 0 & \frac{29}{7} \end{bmatrix} \qquad \det A = (-1)^2\, 29 = 29$$

If we did this right, the determinant should equal 29.

While this makes for a nice definition of determinants, this method of calculation will not be very handy when we are trying to find eigenvalues (except in very special situations). Instead, we will do our calculations using cofactor expansion as in Section 3.1 of the book.

Example

Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Find the eigenvalues of A.

$$\det(A - \lambda I) = 0 \longrightarrow \det\left(\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}\right)$$

$$\begin{matrix} -\lambda & 1 & 1 & -\lambda & 1 \\ 1 & -\lambda & 1 & 1 & -\lambda \\ 1 & 1 & -\lambda & 1 & 1 \end{matrix}$$

$$(-\lambda)^3 + 1 + 1 - (-\lambda) - (-\lambda) - (-\lambda)$$

$$= -\lambda^3 + 3\lambda + 2$$

$$= -(\lambda^3 - 3\lambda - 2)$$

$$= -(\lambda+1)(\lambda-2)(\lambda+1)$$

$$= (\lambda+1)(\lambda^2 - \lambda - 2)$$

$$= (\lambda+1)^2(\lambda - 2)$$

So my eigenvalues are $\lambda = -1, 2$

Can we also find the eigenspaces for this matrix?

$\underline{\lambda = -1}$ $\quad (A + I) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$(A + I)\underline{x} = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad x_1 = -x_2 - x_3$$

$$\begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} :$$

So we get $\operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = (-1)\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Let's look at the definition of a characteristic equation once more. We have $\det(A - \lambda I) = 0$. It turns out that for any $n \times n$ matrix A, $\det(A - \lambda I)$ will always yield a degree n polynomial in the variable λ. We call this the *characteristic polynomial* for A. So to find the eigenvalues for A, we can find the roots of the characteristic polynomial for A.

Example
Suppose A is a 7×7 ~~polynomial~~ matrix with characteristic polynomial $p(\lambda) = (\lambda - 3)^2 \underbrace{(\lambda^2 - 2\lambda + 2)}(\lambda + 5/3)^3$. What are the eigenvalues for A?

$b^2 - 4ac < 0$

$\Rightarrow$ complex roots

Note that the roots 3 and $-5/3$ have different multiplicities in the above polynomial. We say that the *algebraic multiplicity* of an eigenvalue is its multiplicity as a root of the characteristic polynomial. Note also that sometimes the eigenvalues will be complex numbers, but we will not consider such values. The following is an example of a 3×3 matrix for which this occurs (you can check yourself):

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

2. Similarity

Given two $n \times n$ matrices A and B, we say that A is *similar* to B if there exists some invertible matrix P such that $P^{-1}AP = B$, or equivalently $A = PBP^{-1}$. If we let $Q = P^{-1}$ then we see that B is also similar to A.

Theorem If two $n \times n$ matrices are similar, they have the same characteristic polynomial and thus the same eigenvalues.

Proof.
Consider $A - \lambda I$,

$$\begin{aligned} \det(A - \lambda I) &= \det(PBP^{-1} - \lambda I) \\ &= \det(PBP^{-1} - \lambda PP^{-1}) \\ &= \det(P(B - \lambda I)P^{-1}) \\ &= \det P \det(B - \lambda I) \det P^{-1} \\ &= \det(B - \lambda I) \end{aligned}$$

$(\det P \cdot \det P^{-1} = 1)$

So the characteristic polynomials are equal & have the same roots (eigenvalues).

3. Application in Dynamical Systems

See example 5 in section 5.2 of the book.

HW: # 2, 11, 12, 16, 18, 24, 25, 27