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The LU and QR Factorizations

Text Reference: Section 2.5, p. 150

The purpose of this set of exercises is to explore a relationship between two matrix factorizations: the LU factorization and the QR factorization. The first example illustrates a QR factorization. Recall that a QR factorization of an $n\times n$ matrix A is $A=QR$, where R is invertible and upper triangular, and Q has the property that $Q^{\mathrm{T}}Q=I$.

Example: Let $A=\begin{pmatrix} 0 & -2 & 1 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 5 \end{pmatrix}$.

Then $A=QR$, where $Q=\begin{pmatrix} 0 & -\frac{2}{\sqrt{6}} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{2} \end{pmatrix}$ and $R=\begin{pmatrix} \sqrt{2} & 2\sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{6} & -\sqrt{6} \\ 0 & 0 & 2 \end{pmatrix}$.

Question:
1. Confirm that $A=QR$ and that Q and R meet the criteria given above for a QR factorization.

A major question is how Q and R are found. One method is to use something called the Gram-Schmidt process, which is discussed in Section 6.4 of the text. Mathematical and engineering software packages use other techniques which are beyond the scope of an introductory course. The following method is a way to find Q and R which depends only on row reduction.

Step 1: Form the augmented matrix $\left[A^T A \,\middle|\, A^T\right]$ and reduce AA^T to echelon form without using row swapping or scaling.

In the example,

$$\left[A^T A \,\middle|\, A^T\right]=\begin{pmatrix} 2 & 4 & 6 & 0 & 1 & 0 & 1 \\ 4 & 14 & 6 & -2 & 3 & 0 & 1 \\ 6 & 6 & 28 & 1 & 1 & 1 & 5 \end{pmatrix}\sim\begin{pmatrix} 2 & 4 & 6 & 0 & 1 & 0 & 1 \\ 0 & 6 & -6 & -2 & 1 & 0 & -1 \\ 0 & 0 & 4 & -1 & -1 & 1 & 1 \end{pmatrix}$$

Step 2: Take the transpose of the reduced version of A^{T} at the right of the matrix. Call this matrix $\hat{Q}$, and notice that $\hat{Q}^T\hat{Q}$ is a diagonal matrix.

In the example, $\hat{Q}=\begin{pmatrix} 0 & -2 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ and $\hat{Q}^T\hat{Q}=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

Step 3: Let $D = \hat{Q}^T\hat{Q}$, and let D^k be that matrix formed by taking each of the diagonal elements of D to the k$^{\text{th}}$ power. Then Q and R are defined by $Q = \hat{Q}D^{-1/2}$ and $R = D^{-1/2}\hat{Q}A$.

Questions:

2. Show that applying the above formulas to $\hat{Q} = \begin{pmatrix} 0 & -2 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ yields the Q and R presented above.

3. Use this process to find a QR factorization of the following matrices.

a. $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

b. $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & -3 \\ 0 & -2 & 3 \end{pmatrix}$

c. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 0 & 4 \end{pmatrix}$

The reason that this algorithm works involves the LU factorization of Section 2.5. Let A be an $n \times n$ matrix with linearly independent columns. Then it turns out that the matrix $A^{\mathrm{T}}A$ can be row reduced to echelon form without row interchanges. Thus $A^{\mathrm{T}}A$ has an LU factorization $A^{\mathrm{T}}A = LU$. The matrix $\hat{Q}$ is defined by $\hat{Q} = A(L^{-1})^T$.

Question:

4. Use the fact that $A^{\mathrm{T}}A = LU$ to show that $\hat{Q}^T\hat{Q} = U(L^{-1})^T$.

In the example done at the beginning of this exercise set, it happened that $\hat{Q}^T\hat{Q}$ is a diagonal matrix. This happens in general. First, notice that $\left(\hat{Q}^T\hat{Q}\right)^T = \hat{Q}^T\hat{Q}$. This means that the matrix $\hat{Q}^T\hat{Q}$ is **symmetric**; its (i,j) element and its (j,i) element are always equal. But Exercise 4 states that $\hat{Q}^T\hat{Q} = U\left(L^{-1}\right)^T$. It is also known that U is upper triangular and L is lower triangular. Thus L^{-1} is also lower triangular, and $(L^{-1})^{\mathrm{T}}$ must be upper triangular. Since $\hat{Q}^T\hat{Q}$ is the product of two upper triangular matrices, $\hat{Q}^T\hat{Q}$ must also be upper triangular. Finally, since $\hat{Q}^T\hat{Q}$ is both symmetric and upper triangular, $\hat{Q}^T\hat{Q}$ must be a diagonal matrix.

Question:
5. Show that $\hat{Q}^T = L^{-1}A^T$. Why does this outcome show that the row reduction of $\left[A^T A \,\middle|\, A^T\right]$ gives us $\left[U \,\middle|\, \hat{Q}^T\right]$? Hint: recall that $AA^T = LU$. Therefore $\hat{Q}$ may be found by the method above.

Now define $\hat{R} = L^T$ and $D = \hat{Q}^T\hat{Q}$, and adopt the meaning of D^k given above.

Questions:

6. Show that $A = \hat{Q}\hat{R}$.

7. Show that $A=QR$, where $Q = \hat{Q}D^{-1/2}$ and $R = D^{-1/2}\hat{R}$. Also show that $Q^TQ=I$, and that R is an invertible upper triangular matrix.

Thus $A=QR$ is a QR factorization of A. While this method is not generally used to find the QR factorization, it is still interesting to note how the LU factorization was used in proving our result.

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