Section 1.8: Linear Transformations

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Abstract

In this section we want to think very geometrically about the meaning of matrix multiplication: $A\mathbf{x} = \mathbf{b}$ is effectively a transformation of the vector $\mathbf{x}$ into another vector $\mathbf{b}$. So it's transforming the vectors in the space of $\mathbf{x}$, turning them into vectors of the space of $\mathbf{b}$. What kinds of transformations might we be undertaking? That's the subject of this section.

We're going to start by thinking about the spaces as sets of vectors, rather than about coordinates of a vector.

Definition: transformation: a transformation (or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is the **domain** of T, and $\mathbb{R}^m$ is the **codomain**.

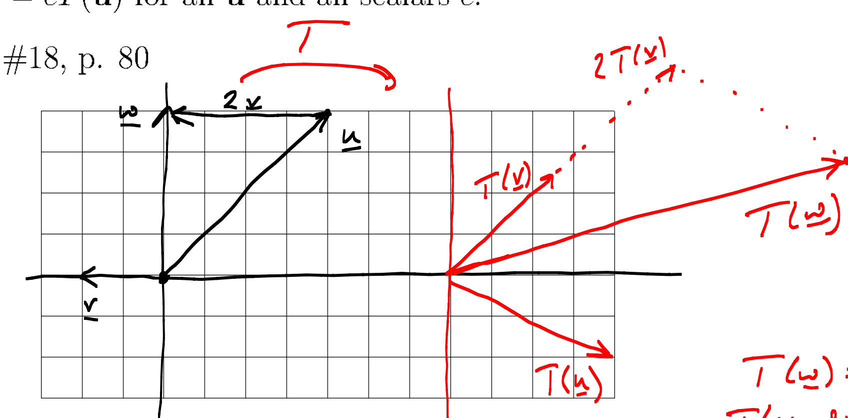
For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ is called the **image** of $\mathbf{x}$ (under the action of T). The set of all images $T(\mathbf{x})$ of vectors $\mathbf{x}$ from the domain is called the **range** of the transformation T.

A transformation T is **linear** if it satisfies the usual conditions:

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in the domain of T
- $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ and all scalars c.

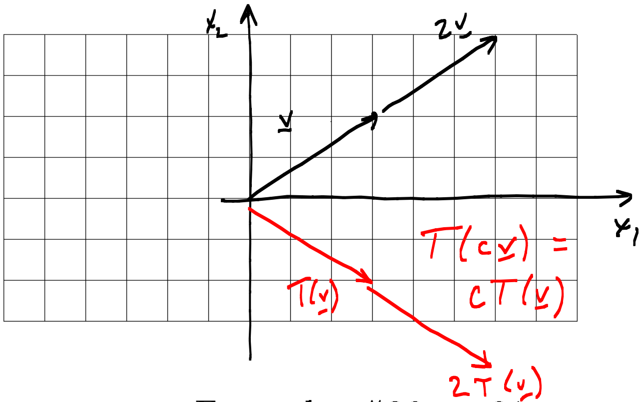
Example: #18, p. 80

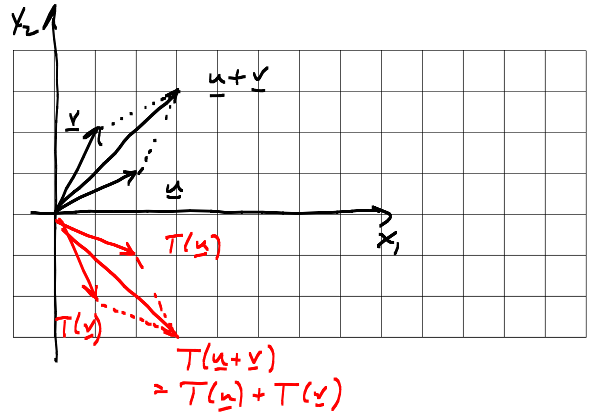
$\underline{w} = \underline{u} + 2\underline{v}$



$T(\underline{w}) = T(\underline{u} + 2\underline{v}) = T(\underline{u}) + T(2\underline{v}) = T(\underline{u}) + 2T(\underline{v})$

Example: #23, p. 81





Example: #26, p. 81

The matrix product $A\mathbf{x}$ represents a linear transformation, as we have already seen. If A is an m x n matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

(a) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$

(b) $A(c\mathbf{u}) = c(A\mathbf{u})$

More generally, a linear transformation satisfies

$$T(c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \ldots + c_pT(\mathbf{v}_p)$$

which is known as the **principle of superposition**.

In this section, several important examples of linear transformation representable by matrices are given, corresponding to

- **projections** (Example 2, p. 76): a dimension is lost, as objects in a higher dimensional space are squashed down into (projected onto) a smaller space.
- **shears** (Example 3, p. 76): vectors are shoved in a direction as though blown by a strong wind.
- **scalings** (Example 4 - contractions and dilations, p. 77): objects are shrunk or stretched, while maintaining their shape.
- **rotations** (Example 5, p. 78): the space is spun about the origin.

As you can well imagine, these sorts of transformations are very useful to the computer scientist, among others: if you want to simulate motion in a computer game, for example, you will be constantly projecting, rotating, and scaling objects. But for translations, computer scientists have need of **affine** transformations, as described in the following examples.

Example: #29, p. 81

Example: #30, p. 81

#34 p81 $\quad T: \mathbb{R}^n \to \mathbb{R}^n$

Suppose T maps $\underline{u}, \underline{v}$ ($\underline{u} \neq \alpha \underline{v}$ for some $\alpha \in \mathbb{R}$ - $\underline{u}$ & $\underline{v}$ are independent) into

$T(\underline{u})$ and $T(\underline{v})$; but $\{T(\underline{u}), T(\underline{v})\}$ is a linearly dependent set. That means $\exists \beta \in \mathbb{R}$ such that $T(\underline{v}) = \beta \cdot T(\underline{u})$ $\Rightarrow$

$$T(\underline{v}) - \beta T(\underline{u}) = \underline{0} \quad \Rightarrow$$

$$T(\underbrace{\underline{v} - \beta \underline{u}}_{= \underline{x} \neq \underline{0}}) = \underline{0}$$

– otherwise we'd have dependence in $\underline{u}$ & $\underline{v}$:

$\underline{v} = \beta \underline{u}$

can't happen.