Problem 1

a. (6 pts) Find the solution set of the following system: $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 1 & 0 \\ 2 & 2 & 1 & 1 \\ 1 & 3 & 1 & 2 \end{bmatrix} \Rightarrow \text{rref} \begin{bmatrix} 1 & 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -\frac{1}{4}x_3 - \frac{1}{4}$

$x_2 = -\frac{1}{4}x_3 + \frac{3}{4}$

x_3 = Free

$$\bar{x} = x_3 \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{4} \\ \frac{3}{4} \\ 0 \end{bmatrix}$$ ✓

b. (4 pts) Give a complete geometric description of the set.

Since this equation only has one free variable, its solution is a line through the point $\begin{bmatrix} -\frac{1}{4} \\ 3/4 \\ 0 \end{bmatrix}$. ✓

Good

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Problem 2

a. (8 pts) Determine all values of h such that the following three vectors are linearly dependent:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 8 \\ -7 \\ h \end{bmatrix}$$

Since no Scalar multiple of v_1 or v_2 will give me v_3 I have to See when they are added together what h will make v_3 in the span of v_1 + v_2.

$$\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} x_1 + \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} x_2 = \begin{bmatrix} 8 \\ -7 \\ h \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 8 \\ 4 & 5 & -7 \\ -7 & 3 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 8 \\ 0 & 13 & -39 \\ 0 & -11 & 56+h \end{bmatrix} \quad \begin{matrix} v_2 = 2 \\ v_2 = -3 \end{matrix}$$

$$-11(-3) = 56+h$$

$$33 = 56+h$$

$$\boxed{-23 = h}$$

b. (2 pts) Describe the geometry in the case of dependence: how does it compare to the geometry in the case of a choice of h leading to independence?

Two vectors are dependent if they are scaler multiples of each other. If there are 3 vectors one vector must lie in the plane generated by the other two vectors. Since the two vectors can generate the whole plane they could also generate the other vector.

If h were any other value then v_3 would not be in the plane generated by v_1 and v_2 and the set would be independent.

Good

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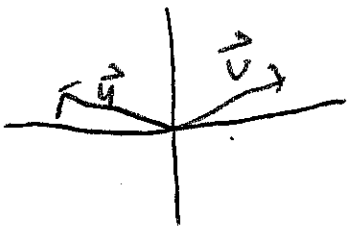
Problem 3 Illustrate each of the following terms or concepts with a well-chosen example:

a. Homogeneous linear system

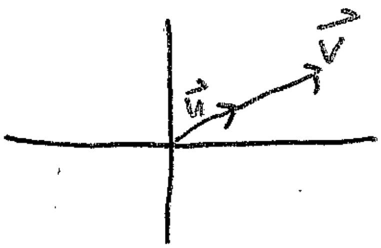
$$Ax = \vec{0} \qquad \begin{bmatrix} 3 & 6 \\ 4 & 7 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

b. span of a set of vectors



span$\{\vec{u}, \vec{v}\}$ is the plane generated by vectors $\vec{u}$, $\vec{v}$ and consists of all linear combinations of $\vec{u}$ and $\vec{v}$

c. linear dependence of a set of vectors



$\vec{v}$ is a scalar multiple of $\vec{u}$ so $\vec{v}$ and $\vec{u}$ are linearly dependent.

d. basic and free variables

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & b \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$x_1, x_2 \to$ basic variables

$x_3 \to$ free variable

e. inconsistent linear system

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -8 \end{bmatrix}$$

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Problem 4 If oxygen and hydrogen are combined, water is produced (and an explosion occurs – but that's another story). ~~The chemical equation looks something~~ like this:

$$2\, H_2 + O_2 \longrightarrow 2 H_2O$$

But there is a problem: there are two oxygen atoms ~~on~~ the left, and only one on the right. We need to balance this equation. ~~Consider each~~ compound as a vector of hydrogen atoms and oxygen atoms, and then

a. Write a linear system that accounts for the conservation of hydrogen and oxygen atoms, making sure that there are the same number on each side of the chemical equation.

$$x_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \checkmark$$

$$\begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}$$

b. Then solve the system to balance the equation (check that your solution works!). How many solutions are possible? What kinds of solutions make sense?

$$\begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \end{bmatrix}$$

Infinate # of solutions but you want the lowest whole number for your solution

So what is that?

-1

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Problem 5 The following figure purports to show three different linear transformations of the space $\Re^2$. In the upper left corner is the unit circle, and the vector $\mathbf{u}$ that rests on one point of the unit circle. The other three panels show purported linear transformations, and the image of the vector $\mathbf{u}$ under each transformation. Your tasks:

a. (7 pts) Decide whether any of the three panels 2-4 could conceivably represent a linear transformation of the plane in panel 1. Could panel 1 represent the result of a linear transformation on itself?

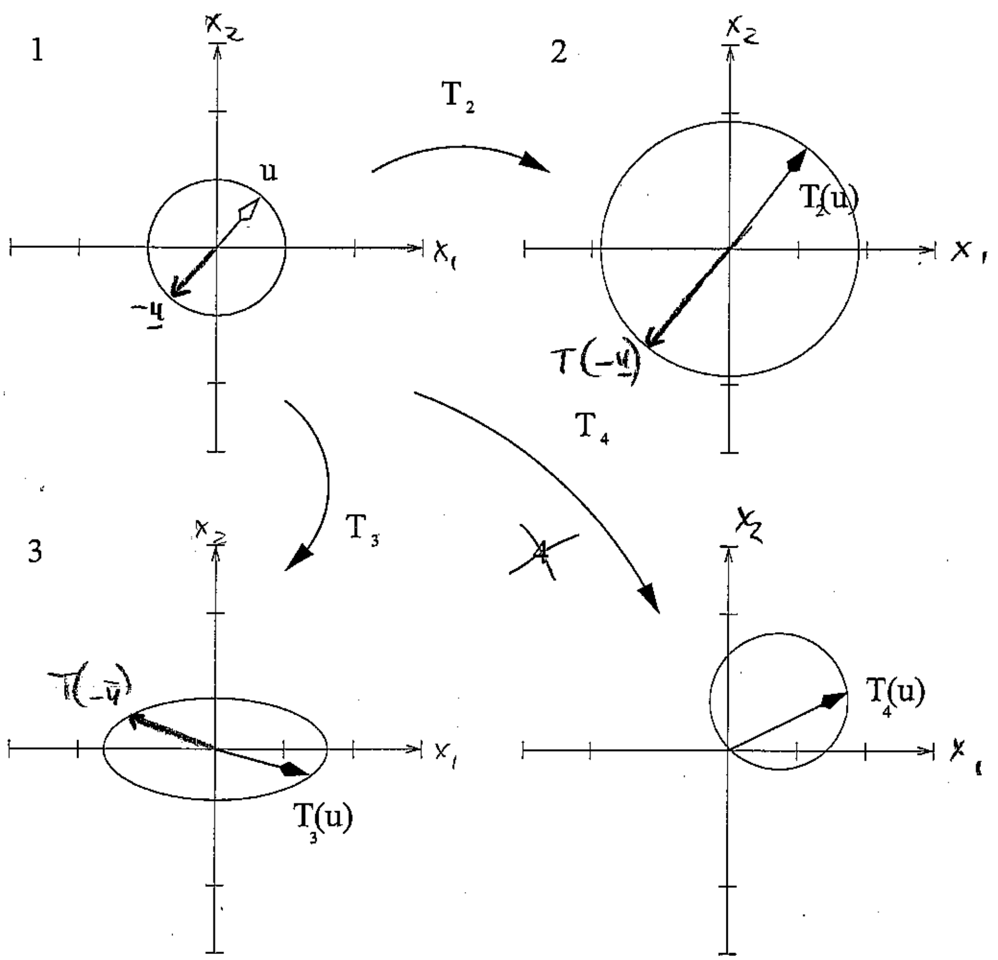
Panel 2: Yes, this is a dilation of the x_1 + x_2 axes. ✓

Panel 3: Yes, this is a dilation of the x_1 axis ✓

Panel 4: No the vector $\bar{u}$ is not mapped inside the circle. ✓

Panel 1: Yes, this could be a rotation of 360°. about the origin ✓

b. (3 pts) Draw the image of $-\mathbf{u}$ under those transformations which are conceivably linear.



Good!

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Problem 6 Consider the following problem, the likes of which you might have encountered in high school:

$$\begin{array}{rcrcrcr} & & 2x_2 & + & 2x_3 & = & 4 \\ 3x_1 & + & 2x_2 & + & 2x_3 & = & 7 \\ x_1 & - & 4x_2 & + & 2x_3 & = & -1 \end{array}$$

Without solving the system, describe at least two distinctly different perspectives that linear algebra provides on the solution of this problem, using the notation and vocabulary of chapter 1 in your descriptions.

a. Perspective 1:

The solution of this problem can be expressed in vector notation, which then in turn allow one to think of the solution geometrically, rather than literally.

$V_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, V_2 = \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}, V_3 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, b = \begin{pmatrix} 4 \\ 7 \\ -1 \end{pmatrix}$. Does b span $\{V_1, V_2, V_3\}$? That is the fundamental question.

Is $\underline{b}$ in the

b. Perspective 2:

We can think of solving this system by Matrix notation, $Ax=b$, where A is the coefficient matrix, b is the matrix on the right-hand side of the equation, and X is the matrix of the X_1, X_2 and X_3 variables.

$$\begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 2 \\ 1 & -4 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$$

✓

c. Perspective 3:

Finally, we can think of of solving the system by simple techniques that we have learned thus far. We can think of this system as a set of linear equations and we can put all the coefficients (and solution) into an augmented matrix.

$\begin{pmatrix} 1 & 2 & 2 & 4 \\ 3 & 2 & 2 & 7 \\ 1 & -4 & 2 & -1 \end{pmatrix}$ and then row reduce.

✓

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Good

Problem 7

a. (8 pts) Now solve the system of Problem 6 using the reduced row echelon form, with partial pivoting, and showing each step. You may scale the rows to get a leading 1 **before** proceeding from row echelon to reduced row echelon form.

$$\begin{bmatrix} 0 & 2 & 2 & 4 \\ 3 & 2 & 2 & 7 \\ 1 & -4 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 2 & 7 \\ 0 & 2 & 2 & 4 \\ 1 & -4 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 2 & 7 \\ 0 & 2 & 2 & 4 \\ 0 & -14/3 & 4/3 & -10/3 \end{bmatrix}$$

pivot again!

$$\sim \begin{bmatrix} 3 & 2 & 2 & 7 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2/3 & 2/3 & 7/3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

b. (2 pts) Why is it unnecessary for you to do partial pivoting, if using infinite precision arithmetic (that is, keeping all decimals of every number in the process)? Why should a computer program to solve general linear systems do partial pivoting?

If you are using infinite precision then there is no risk of losing information by doing calculations on decimal #'s, but computers can only store so many decimal places before it's truncated, partial pivoting allows the computer to lose as little precision as possible.

Good

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