MAT225 Section Summary: 2.2
The Inverse of a Matrix
Summary

The inverse of a matrix is analogous to the multiplicative reciprical: we want to solve $A\mathbf{x} = \mathbf{b}$, and so we'd like to say that $\mathbf{x} = \mathbf{b}/A$ - but we don't know how to say that with matrices! Let's find out....

First of all, this concept only applies when matrices are **square**: so only n x n matrices could possibly be invertible.

Definition: inverse An n x n matrix A is **invertible** if theres exists an n x n matrix C (the **inverse** of A) such that

$$CA = I = AC$$

The inverse C is denoted A^{-1}, and is unique. A square matrix for which the inverse fails to exist is called **singular**.

A simple formula exists for the inverse of a two-by-two matrix: if A is given by

$$A = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right]$$

then, provided $ad - bc \neq 0$,

$$A^{-1} = \frac{1}{ad - bc} \left[\begin{array}{cc} d & -b \\ -c & a \end{array} \right]$$

Otherwise, A is singular. The quantity $ad - bc$ is called the **determinant** of A: $det(A) = ad - bc$.

#1, p. 126 (check!)

Theorem 5: if A is invertible, then $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$: $\mathbf{x} = A^{-1}\mathbf{b}$.

#5, p. 126 (check!)

Theorem 6:

1. If A is invertible, then $(A^{-1})^{-1} = A$.

   (Check #1).

2. If A and B are n x n invertible matrices, then so is AB, and the inverse of AB is the product of the inverses, in the reverse order:

   $$(AB)^{-1} = B^{-1}A^{-1}$$

   More generally, the inverse of a product of any number of invertible matrices is the product of the inverses in reverse order.

#15, p. 126.

3. If A is invertible, then so is A^T, and the inverse of A^T is the transpose of A^{-1}:
$$(A^T)^{-1} = (A^{-1})^T$$

Definition: an **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix. Each elementary matrix is invertible.

If an elementary row operation is performed on an m x n matrix A, the resulting matrix can be written as EA, where the m x m matrix E is created by performing the same row operation on I_m.

#28, p. 127

Theorem 7: n x n matrix A is invertible if and only if A is row equivalent to I_n. The elementary row operations that transform A into I_n simultaneously transforms I_n into A^{-1}.

Theorem 7 suggests a method for finding A^{-1}: row reduce the augmented matrix $[AI_n]$. If A is row equivalent to I_n, then $[AI_n]$ is row equivalent to $[I_nA^{-1}]$.

#1, p. 126

#18, p. 126

#19, p. 126

#21, p. 126

Note : A^{-1} is generally not calculated: we don't need to know its entries to solve $A\mathbf{x} = \mathbf{b}$ (similar to the notion that we don't need to row reduce to reduced row echelon form to solve: we can stop with a triangular matrix).