

## MAT225 Chapter 3 Summary

### Determinants

As a preface, note that we're always talking about square matrices in the following.

#### 1. Theorems/Formulas

**Theorem 2, p. 189:** If  $A$  is a triangular matrix, then  $\det A$  is the product of the entries on the main diagonal of  $A$ .

**Theorem 3, p. 192:** Let  $A$  be a square matrix.

- (a) If a multiple of one row of  $A$  is added to a different row to produce a matrix  $B$ , then  $\det B = \det A$ .
- (b) If two rows of  $A$  are interchanged to produce  $B$ , then  $\det B = -\det A$ .
- (c) If one row of  $A$  multiplied by  $k$  to produce  $B$ , then  $\det B = k \det A$ .

These are the operations of elementary matrices, so we see how to relate the determinant of the matrix  $U$  obtained by row-reduction from  $A$  to the determinant of  $A$  itself:

**Example:** Let's check Theorem 3 against 2 by 2 matrices.

**Formula (1), p. 194:** Suppose a square matrix  $A$  has been reduced to an echelon form  $U$  by row replacements and  $r$  row interchanges. Then

$$\det A = (-1)^r \det U$$

Since  $U$  is triangular, its determinant is simply the product of its diagonal entries.

**Theorem 4, p. 194:** A square matrix  $A$  is invertible if and only if  $\det A \neq 0$ .

**Theorem 5, p. 196:** If  $A$  is an  $n \times n$  matrix, then  $\det A^T = \det A$ .

**Theorem 6, p. 196:** If  $A$  and  $B$  are  $n \times n$  matrices, then  $\det AB = (\det A)(\det B)$ .

## 2. Properties/Tricks/Hints/Etc.

In  $R^n$  the ball of unit volume (that is, a ball centered at the origin of volume 1) is transformed under a linear transformation into an ellipsoid. The volume of the ellipsoid is the absolute value of the determinant. If you like, you can consider that the **definition** of the determinant! See the figure on page 209.

## 3. Summary

We first encountered the determinant when inverting  $2 \times 2$  matrices: it appears in the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The determinant is denoted  $\det A$ , and  $\det A = ad - bc$ . Obviously if  $\det A = 0$ , then the  $2 \times 2$  matrix is not invertible, so the determinant was mixed up with the idea of invertibility.

The determinant was important classically, perhaps more so than it is today. The idea of cofactors is classical, elegant, but not particularly practical. As mentioned in the text, the calculation of a determinant is carried out by the method of  $LU$  decomposition, and relies upon the simple fact that the determinant of a triangular (and especially diagonal) matrix is the product of the diagonal elements.

This is certainly the most important fact: the determinant represents the volume of the image of the ball of unit volume under the linear transformation represented by  $A\mathbf{x}$ . So if the determinant is zero, the ball's image has been "squashed" so that it has zero volume. This

means that the matrix is singular, and cannot be inverted. This is the key fact, and the fact that we will encounter again when we bump up against **eigenvalues**.