MAT225 Section Summary: 4.2
Null spaces, column spaces, and linear transformations
Summary

The solution set of the homogeneous equation $A_{m \text{ x } n}\mathbf{x} = \mathbf{0}$ forms a subspace of $\mathbb{R}^n$, as one can see easily:

1. the zero vector is in the solution set (the trivial solution);
2. Consider two vectors in the solution set, $\mathbf{u}$ and $\mathbf{v}$: then $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$, so the solution set is closed under addition.
3. Consider a vectors in the solution set, $\mathbf{u}$ and an arbitrary constant c: then $A(c\mathbf{u}) = cA\mathbf{u} = \mathbf{0}$, so the solution set is closed under scalar multiplication.

Null space of an m x n matrix A: the null space of an m x n matrix A, denoted Nul A, is the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{b}$. It is the set of all $\mathbf{x} \in \mathbb{R}^n$ that are mapped to the zero vector of $\mathbb{R}^m$ by the transformation $\mathbf{x} \longrightarrow A\mathbf{x}$.

Theorem 2: The null space of an m x n matrix A is a subspace of $\mathbb{R}^n$.

Example: #3, p. 234.

Notice that the number of vectors in the spanning set for Nul A equals the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.

Column space: Another subspace associated with the matrix A is the column space, Col A, defined as the span of the columns of A: Col A = Span $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. As a span, it is clearly a subspace (Theorem 3).

Col $A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x}$ for some $\mathbf{x}$ in $\mathbb{R}^n\}$, which says that Col A is the range of the transformation $\mathbf{x} \longrightarrow A\mathbf{x}$.

Example: #16, p. 234

The null space lives in the row space of the matrix A, and the column space lives in the column space of A.

Example: #22, p. 235

Linear Transformation: A linear transformation T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

1. $\mathrm{T}(\mathbf{u} + \mathbf{v}) = \mathrm{T}(\mathbf{u}) + \mathrm{T}(\mathbf{v})$

2. T(c$\mathbf{u}$) = cT($\mathbf{u}$)

The **kernel** (or **null space**) of T is the set of $\mathbf{u}$ such that $T(\mathbf{u}) = \mathbf{0}$. The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V.

Example: #30, p. 235

Examples of linear transformations include matrix transformations, as well as differentiation in the vector space of differentiable functions defined on an interval (a, b).

Example: #33, p. 235