MAT225 Section Summary: 6.5
Least-Squares Problems
Summary

Okay! This is it: the section with the formula for the solution of the least-squares problem, which is known as the linear regression problem in statistics. This is how we find a nice fit to linear (and specialized types of non-linear) models. What an amazingly powerful tool this is, and it's based on some simple linear algebra....

least-squares solution: If $A_{m \text{ x } n}$ and $\mathbf{b}$ is in $\mathbb{R}^m$, then a least-squares solution of $A\mathbf{x} = \mathbf{b}$ is $\hat{\mathbf{x}}$ in $\mathbb{R}^n$such that

$$\|\mathbf{b} - A\hat{\mathbf{x}}\| \leq \|\mathbf{b} - A\mathbf{x}\|$$

for all $\mathbf{x}$ in $\mathbb{R}^n$.

Q: Take a look at that equation above, and tell me where the name "least-squares" comes from....

Now, consider the projection of $\mathbf{b}$ onto the Col A,

$$\hat{\mathbf{b}} = \text{ proj}_{\text{Col } A}\mathbf{b}$$

and let $\hat{\mathbf{x}}$ be defined as the solution of

$$A\hat{\mathbf{x}} = \hat{\mathbf{b}}.$$

Q: How do we know that there **is** such a solution?

We know that $\mathbf{b} - \hat{\mathbf{b}}$ is orthogonal to Col A, so

$$A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}.$$

from which we arrive at

$$A^TA\hat{\mathbf{x}} = A^T\mathbf{b}.$$

Hence $\hat{\mathbf{x}}$ is a solution of the equation

$$A^TA\mathbf{x} = A^T\mathbf{b}$$

(the so-called **normal equations**). There may be many (in infinite number!) of solutions of the normal equations.

Theorem 13: The set of least-squares solutions of $A\mathbf{x} = \mathbf{b}$ coincides with the nonempty set of solutions of the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$.

However, if $A^T A$ is invertible, then the solution is unique:

Theorem 14: The matrix $A^T A$ is invertible $\iff$ the columns of A are linearly independent. In this case, the equation $A\mathbf{x} = \mathbf{b}$ has only one least-squares solution $\hat{\mathbf{x}}$, and it is

$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}.$$

Problems:

1. #2, p. 416

2. #5, p. 416

3. #13, p. 416

4. #23, p. 417

5. #24, p. 417

6. #25, p. 417