Section 1.4: Predicate Logic

January 28, 2008

Abstract

We now consider the logic associated with predicate wffs, including a new set of derivation rules for demonstrating validity (the analogue of tautology in the propositional calculus).

1 Derivation rules

- First of all, all the rules of propositional logic still hold. Whew! Propositional wffs are simply boring, variable-less predicate wffs.

- Our author suggests the following "general plan of attack":

  - strip off the quantifiers

  - work with the separate wffs

  - insert quantifiers as necessary

  Now, how may we legitimately do so?

- New rules for predicate logic: in the following, you should understand by the symbol x in $P(x)$ an expression with free variable x, possibly containing other (quantified) variables: e.g.

$$P(x) = (\forall y)(\exists z)Q(x, y, z) \tag{1}$$

  - **Universal Instantiation**: from $(\forall x)P(x)$ deduce $P(t)$.

    Caveat: t must not already appear as a variable in the expression for $P(x)$: in the equation above, (1), it would not do to use $P(y)$ or $P(z)$, as they appear in the expression (in a quantified fashion) already.

    Example: Practice 22, p. 48/51

Prove:

$$(\forall x)[P(x) \to R(x)] \wedge [R(y)]' \to [P(y)]'$$

1. $(\forall x)[P(x) \to R(x)]$ hyp
2. $[R(y)]'$ hyp
3. $P(y) \to R(y)$ 1, ui
4. $[P(y)]'$ 3, 2, mt

– **Existential Instantiation**: from $(\exists x)P(x)$ deduce $P(t)$.

Caveat: t must be introduced for the first time (so do these early in proofs). You can do a universal instantiation which also uses t after an existential instantiation with t, but not *vice versa* (e.g. Example 27).

Generally existentials before universals.

Example: Ex. #11, p. 58/61 (start).

Prove that the following wff is a valid argument:

$$(\forall x)P(x) \wedge (\exists x)Q(x) \to (\exists x)[P(x) \wedge Q(x)]$$

1. $(\forall x)P(x)$ hyp
2. $(\exists x)Q(x)$ hyp
3. $Q(t)$ 2, ei
4. $P(t)$ 1, ui
5. $P(t) \wedge Q(t)$ 3, 4, conj
6. $(\exists x)[P(x) \wedge Q(x)]$ 5, eg.

[from Existential Generalization, below]

– **Universal Generalization**: from $P(x)$ deduce $(\forall x)P(x)$.

Caveats:

* $P(x)$ hasn't been deduced by existential instantiation from any hypothesis in which x was free, and

* $P(x)$ hasn't been deduced by existential instantiation from another wff in which x was free. For example, suppose that we wanted to prove, in the domain of the integers, that:

$$(\forall x)(\exists y)(x + y = 0) \rightarrow (\forall x)(x + a = 0)$$

	1.$(\forall x)(\exists y)(x + y = 0)$	*hyp*
	2.$(\exists y)(x + y = 0)$	$1, ui$ ← wff in which x was free
$P(x) \rightarrow$	3.$x + a = 0$	$2, ei$
	4.$(\forall x)(x + a = 0)$	3, **incorrect ug**

Example: Ex. #17, p. 58/62

Prove:

$$(\forall x)[P(x) \rightarrow Q(x)] \rightarrow [(\forall x)P(x) \rightarrow (\forall x)Q(x)]$$

1. $(\forall x)[P(x) \rightarrow Q(x)]$ — hyp
2. $(\forall x)P(x)$ — deduction method
3. $P(x) \rightarrow Q(x)$ — 1, ui
4. $P(x)$
5. $Q(x)$ — 3, 4, mp
6. $(\forall x)Q(x)$ — 5, ug

(Note: the deduction method still applies, of course.)

– **Existential Generalization**: from $P(a)$ deduce $(\exists x)P(x)$.

  Caveat: x must not appear in $P(a)$.

  Example: Ex. #11, p. 58/61 (finish).

2 Some results/notes

– Note that

$$(\forall y)[P(x) \rightarrow Q(x, y)] \iff [P(x) \rightarrow (\forall y)Q(x, y)]$$

as shown on pp. 51-52/55, and

$$(\exists y)[P(x) \to Q(x,y)] \iff [P(x) \to (\exists y)Q(x,y)]$$

as Gersting suggests. How would we demonstrate that? (See "temporary hypothesis", below.)

This means that we can "pass over" predicates outside our own scope, or include them within our own scope. This is similar to what we do with summation notation, when, for example, we can write

$$\sum_{i=1}^{m}\sum_{j=1}^{n} A(i)B(j) = \sum_{i=1}^{m} A(i) \sum_{j=1}^{n} B(j)$$

– Note also the **method** of proof: the author introduces a **temporary hypothesis**. If you think about the deduction method, it takes a conclusion which is an implication and rewrites it so that the implication disappears (the antecedent becomes one of the hypotheses). Similarly, we can take an hypothesis (in this case, one which we introduce) and turn a conclusion into an implication. This is the deduction method backwards! That is, suppose that one starts with $P(x)$ as true. You add a "temporary hypothesis" $Q(x)$, and from that deduce $R(x)$:

$$P(x) \wedge Q(x) \to R(x)$$

Using the deduction method backwards, we conclude that

$$P(x) \to (Q(x) \to R(x))$$

Since $P(x)$ implies the implication $Q(x) \to R(x)$, we can add it as an hypothesis to our argument:

$$P(x) \wedge (Q(x) \to R(x))$$

Think about it....

Look at the three proofs using a temporary hypothesis (Examples #31, and 32(a,b)). Notice how the introduction of the temporary hypothesis ends with an implication, which is then useful for the continuation of the proof.

Example: Practice 25, p. 52/56

Prove:

$$(\forall x)[(B(x) \vee C(x)) \to A(x)] \to (\forall x)[B(x) \to A(x)]$$

1.	$(\forall x)[B(x) \vee C(x) \to A(x)]$	hyp
1.5	$B(x) \vee C(x) \to A(x)$	1, ui
2.	$B(x)$	tmp hyp
3.	$B(x) \vee C(x)$	2, add
4.	$A(x)$	1.5, 3, mp
5.	$B(x) \to A(x)$	tmp hyp discharged

6. $(\forall x)(B(x) \to A(x))$ 5, ug.

So now, how would we demonstrate that

$$(\exists y)[P(x) \to Q(x,y)] \iff [P(x) \to (\exists y)Q(x,y)]$$

$\leftarrow$ (#23/24, *)

$\rightarrow$

Example: #31/34

Every computer science student works harder than somebody, and everyone who works harder than any other person gets less sleep than that person. Maria is a computer science student. Therefore, Maria gets less sleep than someone else. $C(x), W(x,y), S(x,y), m$

$(\forall x)[C(x) \to (\exists y)[W(x,y)]] \wedge$

$(\forall x)(\forall y)[W(x,y) \to S(x,y)] \wedge$

$C(m) \to$

$(\exists y)[S(m,y)]$

$S(x,y)$ - x gets less sleep than y

$W(x,y)$ - x works harder than y

1. $(\forall x)[C(x) \to (\exists y)[W(x,y)]]$ hyp
2. $(\forall x)(\forall y)[W(x,y) \to S(x,y)]$ "
3. $C(m)$ "
4. $C(m) \to (\exists y) W(m,y)$ 1, ui
5. $(\exists y) W(m,y)$ 3,4, mp
6. $W(m,n)$ 5, ei
7. $W(m,n) \to S(m,n)$ 2, ui twice
8. $S(m,n)$ 6,7, mp
9. $(\exists y) S(m,y)$ 8, eg

Example #31 $\quad [P(x) \to (\forall y)\, Q(x,y)] \to (\forall y)[P(x) \to Q(x,y)]$

1. $P(x) \to (\forall y)\, Q(x,y)$ hyp
2. $P(x)$ tmp hyp
3. $(\forall y)\, Q(x,y)$ 1, 2, mp
4. $Q(x,y)$ 3, ui

$(P(x) \to \forall y\, Q(x,y)) \wedge P(x) \to Q(x,y)$

so - deduction method backwards

$[P(x) \to \forall y\, Q(x,y)] \to (P(x) \to Q(x,y))$

5. $P(x) \to Q(x,y)$ tmp hyp discharged
6. $(\forall y)\,(P(x) \to Q(x,y))$ (we now assume this as a new hypothesis, since it follows from 1.)