Section 2.1: Proof Techniques

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Abstract

Sometimes we see patterns in nature and wonder if they hold in general: in such situations we are demonstrating inductive reasoning to propose a **theorem**, which we can attempt to prove via deductive reasoning. From our work in Chapter 1, we conceive of a theorem as an argument of the form $P \rightarrow Q$, whose validity we seek to demonstrate.

This section outlines a variety of proof techniques, including direct proofs, proofs by contraposition, proofs by contradiction, proofs by exhaustion, and proofs by dumb luck or genius! You have already seen each of these in chapter 1 (with the exception of "dumb luck or genius", perhaps!).

1   Theorems and Informal Proofs

The theorem-forming process is one in which we

- make observations about nature, about a system under study, etc.;

- discover patterns which appear to hold in general;

- state the rule; and then

- attempt to prove it (or disprove it!).

This process is formalized in the following definitions:

- **inductive reasoning** - drawing a conclusion based on experience, which one might state as a conjecture, hypothesis, or theorem.

- **deductive reasoning** - application of a logic system to investigate a proposed conclusion based on hypotheses (hence proving, disproving, or, failing either, holding in limbo the conclusion).

- **counterexample** - an example which violates a proposed rule (or theorem), proving that the rule doesn't work in the particular interpretation.

Before attempting to prove a theorem, we should be convinced of its correctness; if we doubt it, then we should pursue the line of our doubt, and attempt to find a counterexample.

1.1 Exhaustive Proof

- **Example: The Four-color problem**

  - Description (see p. 369/432).
  - This theorem is partly famous because it provided the first example of a computer-aided proof of a major result. The reason the computer became useful was that the proof came down to testing a rather large number of special cases (proof by exhaustion).

  When there are only a few things (in particular, a finite number) to test, we can use proof by exhaustion.

- **Example: Prolog**, in which we use Horn Clauses, resolution, and a finite number of possibilities (finite database) to decide if a theorem is true.

- **Example: My young friend Sam**

  Kids are wonderful at developing conjectures, and sometimes even applying deductive logic, as illustrated in my friend Sam's Story. Sam made an amazing application of proof by exhaustion.

  Kids will make all sorts of false conjectures (e.g. "All animals living in the ocean are fish," or "all meat-eaters are animals"), and parents, siblings, friends, and teachers all have the priviledge and pleasure of coming up with counterexamples.

1.2 Direct Proof

The most obvious and common technique is the direct proof: you start with your hypotheses $\{P_i\}$, and proceed directly toward your conclusion Q:

$$P_1 \wedge P_2 \wedge \cdots \wedge P_n \to Q$$

Example: Exercise 9/12, p. 92/98 Prove directly that the sum of even integers is even.

$(\forall x)(\forall y)(x \text{ even } \wedge \text{ } y \text{ even } \Rightarrow x+y \text{ even})$

1. x even   hyp

2. y even   hyp

3. $x = 2 \cdot m$ where $m \in \mathbb{Z}$ defn

7. $m+n \in \mathbb{Z}$ closure

8. $x+y$ even   6,7, defn

4. $y = 2 \cdot n$ " $n \in \mathbb{Z}$ "   evenness
5. $x + y = 2n + 2n$ addition !
6. $x + y = 2(m+n)$ dist. prop.

1.3   Contraposition

If $P \rightarrow Q$ isn't getting you anywhere, you can use your logic systems
to rewrite it as $Q' \rightarrow P'$ (the contrapositive). This is called "proof by
contraposition".

Example: Practice 4 and 5, p. 89/94:   The statements from

chapter 1.1 are:

(a) If the rain continues, then the river will flood.

Converse :
- If the river floods, that the rain continued.
- "If the river will flood, then the rain continues."
- If the river did not flood, then the rain did not
                                                        continue.

(b) A sufficient condition for network failure is that the central switch
    goes down.

Converse :
    If the network fails, then the central switch has
                            gone down.

Contrapositive :
    If the network's up, the central switch is
                                            fine.

(c) The avocados are ripe only if they are dark and soft.

(d) A good diet is a necessary condition for a healthy cat.

Example: Exercise 17/21, p. 93/99   Prove: If a number x is
positive, so is $x + 1$.   (Domain : real numbers.)
            $(\forall x)(x \text{ positive} \rightarrow x+1 \text{ positive})$

1. x positive hyp	$(\forall x)(x+1 \text{ not positive} \rightarrow$
2. $x > 0$ defn	$x \text{ not positive })$
	1. $x+1$ not positive hyp
	2. $x+1 \leq 0$ defn
	3. $x \leq -1$ add prop of inequal'y
	4. x not positive 3, defn.

1.4 Contradiction

$$Q$$
$$P \rightarrow (Q' \rightarrow 0)$$

Contradiction represents some interesting logic: again, we want to prove $P \rightarrow Q$, but rather than proceed directly, we seek to demonstrate that $P \wedge Q' \rightarrow 0$: that is, that P and Q' leads to a contradiction. Then we cannot have both P true, and Q false - which would lead to $P \rightarrow Q$ false, of course.

Example: Exercise 22/26, p. 93/99 Prove: If x is an even prime number, then $x = 2$.

$(\forall x)(x \text{ even} \wedge x \text{ prime} \rightarrow x = 2)$

1. x even   hyp
2. x prime   hyp
3. $x \neq 2$ (trying a proof by contradiction)
4. $x = 2m$, $m \in \mathbb{Z}$ 1, defn. of even
5. x is divisible only by itself   2, defn. of prime.
   and 1
6. $2 \backslash x$ (2 divides x)   defn of divisibility
7. $x = 2$ 5, $\leftarrow$ $2 \neq 1$!
8. $x = 2 \wedge (x = 2)'$ 4,7 contradiction

$\therefore x = 2$ by contradiction

Table 1: Summary of useful proof techniques, from Gersting, p. 91/96.

Proof Technique	Approach to Prove $P \rightarrow Q$	Remarks
Exhaustive Proof	Demonstrate $P \rightarrow Q$ for all examples/cases.	Examples/cases finite
Direct Proof	Assume P, deduce Q.	Standard approach
Contraposition	Assume Q', deduce P'.	Q' gives more ammo?
Contradiction	Assume $P \wedge Q'$, deduce contradiction.	

1.5 Serendipity

Mathematicians often spend a great deal of time finding the most "elegant" proof of a theorem, or the shortest proof, or the most intuitive proof. We may stumble across a beautiful proof quite by accident ("serendipitously"), and those are perhaps the most pleasant proofs

of all. There is a wonderful story associated with Exercise 55/69, p. 94/100.

Prove: the sum of the integers from 1 to 100 is ~~5100~~ 5050.

$$1 + 2 + 3 + \cdots + 98 + 99 + 100$$
$$100 + 99 + 98 + \cdots + 3 + 2 + 1$$

$$\frac{100 \cdot 101}{2} = 5050$$

$$\sum_{i=1}^{n} i = n\frac{(n+1)}{2} \quad \text{Conjecture (based on limited evidence ...)}$$

Inductive reasoning ...