Section 2.2: Induction

February 6, 2008

Abstract

In this section we investigate a powerful form of proof called **induction**. This is useful for demonstrating that a property, call it $P(n)$, holds for all integers n greater than or equal to 1.

Actually, the "1" above is not essential: any "base integer" will do (like 0, for example: it really only matters that there be a "ground floor", or "anchor").

1 Induction

Induction is a very beautiful and somewhat subtle method of proof: the idea is that we want to demonstrate a property associated with natural numbers (or a subset of the natural numbers). As a typical example, consider a theorem of the following type (which we might call "Gauss's theorem," hypothesized when he was seven or so):

Prove that the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

An induction proof goes something like this:

- We'll show that it's true for the first case (usually $k = 1$, called the base case). While the first case is often $k = 1$, this isn't mandatory: we simply need to be sure that there is a first case for which the property is true. $k = 0$ is another popular choice....
- Then we'll show that, if the property is true for the k^{th} case, then it's true for the $(k + 1)^{th}$ case (the inductive step).
- Then we'll put them together: if it's true for 1, then it's true for 2; if it's true for 2, then it's true for 3; .... "to infinity, and beyond!" Or up the ladder, as our author would say.

Imagine dominoes falling. That's what it's like.

The most commonly used form of the principle of induction is expressed as follows:

First Principle of Mathematical Induction:

$$\left.\begin{array}{l} 1.\ P(1) \text{ is true} \\ 2.\ (\forall k)[P(k) \text{ true } \rightarrow P(k+1) \text{ true }] \end{array}\right\} \rightarrow P(n) \text{ true for all positive integers } n$$

or, more succinctly,

$$\boxed{P(1) \wedge (\forall k)[P(k) \rightarrow P(k+1)] \rightarrow (\forall n)P(n)}$$

where the domain of the interpretation is the natural numbers. This is just *modus ponens* applied over and over again. Put *modus ponens* into an infinite loop, because we want it to run off to infinity! This might be the first infinite loop you've ever liked!

Vocabulary:

- **inductive hypothesis**: $P(k)$
- **basis step** (base case, anchor): establish $P(1)$
- **inductive step** (implication): $P(k) \rightarrow P(k+1)$

Example: (Practice 7, or "Gauss's theorem") Prove that, for any natural number n, $\underbrace{1+2+3+\ldots+n = \frac{n(n+1)}{2}}_{P(n)}$.

① Anchor : $n=1$ sum of the first one natural numbers is 1, which is equal to $\frac{1\cdot(1+1)}{2}$ ✓

② ⟹ : Assume $P(k)$, & consider $P(k+1)$:

$$1+2+3+\ldots+k+(k+1) = \frac{(k+1)((k+1)+1)}{2}$$

know this by $P(k)$

$$\underbrace{1+2+3+\ldots+k}_{\frac{k(k+1)}{2}} + (k+1) = \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$ ✓

$$= \frac{(k+1)((k+1)+1)}{2}$$

$\therefore (\forall n)P(n)$

Example: Exercise 34, p. 106/114: Prove that $2^{n-1} \leq n!$ for $n \geq 1$. By induction,

① Anchor : show $P(1)$: $2^{1-1} \leq 1!$

$$2^{1-1} = 2^0 = 1 = 1!$$ ✓

② ⟹ : Assume $P(k)$, & consider $P(k+1)$:

$$2^{(k+1)-1} \leq (k+1)!$$

$$2^{(k+1)-1} = 2^k = 2 \cdot 2^{k-1} \leq 2 \cdot k! \leq (k+1)k!$$

$= (k+1)!$ (for $k \geq 1$)

$\leq (k+1)!$ ✓

$\therefore (\forall n) P(n)$: $\boxed{\forall n \in \mathbb{N},\ 2^{n-1} \leq n!}$

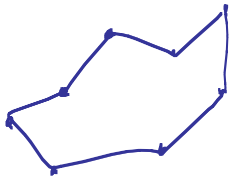
A second (and seemingly more powerful) form of induction is given by the **Second Principle of Mathematical Induction**:

$$\left.\begin{array}{l} 1.\ P(1) \text{ is true} \\ 2.\ (\forall k)[P(r) \text{ true for all } r, \\ \qquad 1 \leq r \leq k \rightarrow P(k+1) \text{ true }] \end{array}\right\} \rightarrow P(n) \text{ true for all positive integers } n$$

This principle is useful when we cannot deduce $P(k+1)$ from $P(k)$ (for k alone), but we can deduce $P(k+1)$ from all preceeding integers, beginning at the base case.

Example: Exercise 64/66b, p. 109/116.

To prove: the sum of interior angles of an n-sided simple closed polygon is $(n-2)\,180°$, for $n \geq 3$,

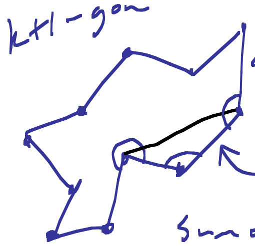


"n-gon"

① Anchor: $n=3$. Is the sum of interior 180° for a 3-gon? ✓

② ⇒ Assume $P(r)\ \forall r \in \{3, \ldots, k\}$; consider $P(k+1)$: the sum of interior angles of k+1-gon is $(k+1-2)\,180° = (k-1)\,180°$

k+1-gon



k-gon – $P(k) \rightarrow$ sum of interior angles is $(k-2)\,180°$

3-gon – $P(3) \rightarrow$ sum $= 180°$

Sum of interior angles for the k+1-gon is the " " " " " " k-gon & 3-gon

$$\text{Sum} = (k-2)180 + 180 = (k-1)180$$
$$= ((k+1)-2)180$$

✓

In spite of appearances, these two principles are equivalent; furthermore they are also equivalent to the **Principle of Well-Ordering**, which states that every collection of positive integers that contains any members at all has a smallest member.

Example: Prove that the first principle of induction implies well-ordering.

① Anchor : sets of a single element have a smallest element - the only element!

② ⇒ : Assume that sets of k elements have a smallest element (P(k)). Demonstrate P(k+1).

Partition a set with k+1 elements into two subsets : of k elements + of 1 element. The k-element subset has a smallest element, + the 1-element subset has a smallest element; put the smallest elements into a two-element set. (to bottom)

A Couple of Fun Examples:

(a) The prisoner's last request (finite backwards induction!)

(b) Now that we understand induction, let's use it to prove an amazing fact: All horses are the same color.

Proof: By induction, on the number of horses.

Base case: 1 horse. No problem! Same color.

Inductive step: we'll show that if it is true for any group of N horses, that all have the same color, then it is true for any group of $N+1$ horses.

Well, given any set of N+1 horses, if you exclude the last horse, you get a set of N horses. By the inductive step these N horses all have the same color. But by excluding the first horse in the pack of N+1 horses, you can conclude that the last N horses also have the same color. Therefore all N+1 horses have the same color.

QED – or have we?

(from above)

We need $P(2)$ as well. Demonstrate that given two elements of $\mathbb{N}$, one is smaller (by the order of the natural numbers), & that this smallest element (call it s) is the smallest element of the $k+1$ set:

$$s \le s_1 \le e_i \quad \text{for all } i = 1, \dots, k$$

$$s \le s_2 = e_{k+1}$$

So s is the smallest element of the $k+1$ set. ✓

Sometimes you need an extra base case or two (as we'll see in recursion). The same problem here (of requiring $P(2)$) plays a role in the "paradoxical" fact that all horses are the same color....