Section 2.4 (2.4/2.5): Recursion and Recurrence Relations

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Abstract

In this section we examine multiple applications of recursive definition, and encounter many examples. Recurrence relations are defined recursively, and solutions can sometimes be given in "closed-form" (that is, without recourse to the recursive definition). We will solve one type of linear recurrence relation to give a general closed-form solution, the solution being verified by induction.

1 Recursion

A **recursive definition** is one in which

1. A basis case (or cases) is given, and

2. an inductive or recursive step describes how to generate additional cases from known ones.

Example: the Factorial function sequence:

1. $F(0) = 1$, and $\quad 0! = 1$

2. $F(n) = nF(n-1)$. $\quad n! = n(n-1)!$

Note: This method of defining the Factorial function obviates the need to "explain" the fact that $F(0) = 0! = 1$. For that reason, it's better than defining the Factorial function as "the product of the first n positive integers," which it is from $n = 1$ on....

In this section we encounter examples of several different objects which are defined recursively (See Table 2.5, p. 131/139):

- **sequences** – an enumerated list of objects (e.g. Fibonacci numbers - Practice 12, p. 122/130 - history, #32/34, p. 142/143)

  I'm very fond of lisp:

  1, 1, 2, 3, 5, 8, 13, 21, 34, ⋮

```
(defun fib(n)
  (case n
        (0 1)
        (1 1)
        (t (+ (fib (- n 1)) (fib (- n 2))))
        )
  )
> (fib 4)
5
> (mapcar #'fib (iseq 0 8))
(1 1 2 3 5 8 13 21 34)
```

  Note: The differences in examples #31 and #32 illustrate why you want to stop and think before you attempt a proof!

- **sets** (e.g. finite length and palindromic strings - Example 34 and Practice 16 and 17, pp. 124-125/133)

Practice 17: $A = \{0, 1\}$

Define palindromic strings on this A.

Base case(s): λ, "0", & "1" are palindromes

Recursion: If p is a palindrome, then apa is as well, where a is another palindrome.

Examples:

0, 1, 00, 11, 101,

- **operations** (e.g. string concatenation - Practice 18, p. 126/134)

- **algorithms** (e.g. BinarySearch - Practice 20, p. 131/139; check out Example #41, p. 130/139, for the definition of "middle".)

  Or my favorites, such as unix shell scripts. Here's one one might call "recurse", for applying an operations to all "ordinary" files:

```
#!/bin/sh
command=$1
files=`ls`
for i in $files
do
        if test -d $i
        then
            cd $i
            directory=`pwd`
            echo "changing directory to $directory..."
            recurse "$command"
            cd ..
        elif test -h $i
        then
            echo $i is a symbolic link: unchanged
else
            $command $i
        fi
done
```

2 Solving Recurrence Relations

Vocabulary:

- **linear recurrence relation:** $S(n)$ depends linearly on previous $S(r)$, $r < n$:

$$S(n) = f_1(n)S(n-1) + \cdots + f_k(n)S(n-k) + g(n)$$

The relation is called **homogeneous** if $g(n) = 0$. (Both Fibonacci and factorial are examples of homogeneous linear recurrence relations.)

- **first-order:** $S(n)$ depends only on $S(n-1)$, and not previous terms. (Factorial is first-order, while Fibonacci is second-order, depending on the two previous terms.)

- **constant coefficient:** In the linear recurrence relation, when the coefficients of previous terms are constants. (Fibonacci is constant coefficient; factorial is not.)

- **closed-form solution**: $S(n)$ is given by a formula which is simply a function of n, rather than a recursive definition of itself. (Both Fibonacci and factorial have closed-form solutions.)

The author suggests an "expand, guess, verify" method for solving recurrence relations.

Example: The story of T

1. Practice 11, p. 121/130

$T(1) = 1$

$T(n) = T(n-1) + 3$ for $n \geq 2$.

Expand:

$T(1) = 1$
$T(2) = 4$
$T(3) = 7$
$T(4) = 10$
$T(5) = 13$

closed form soln.

$\boxed{T(n) = 3n - 2}$

$T(1) = 1$ ✓

$T(2) = 4$ ✓

linear growth.

2. Practice 19, p. 128/137: Here is the recurrence relation for Example 11, p. 121/130, in lisp:

```
(defun Tee(n)
  (if (integerp n)
      (cond
       ((>= n 2)
        (+ (Tee (- n 1)) 3)
        )
       ((= n 1)
        1
        )
       (t (print "Tilt! Only positive ints allowed..."))
       )
    (print "Tilt! Only positive ints allowed...")
    )
  )
> (tee 2)
4
> (mapcar #'tee (iseq 1 10))
(1 4 7 10 13 16 19 22 25 28)
```

3. Practice 21, p. 133/148

Example: general linear first-order recurrence relations with constant coefficients.

$$S(1) = a$$
$$S(n) = cS(n-1) + g(n)$$

"Expand, guess, verify" (then prove by induction!):

$$\boxed{S(n) = c^{n-1}S(1) + \sum_{i=2}^{n} c^{n-i}g(i)}$$

objective: closed-form soln.

Expand

$S(1) = a$

$S(2) = c \cdot S(1) + g(2)$

$S(3) = c \cdot S(2) + g(3) = c(c \cdot S(1) + g(2)) + g(3)$

$= c^2 \cdot S(1) + c \cdot g(2) + g(3)$

$S(4) = c \cdot S(3) + g(4) = c[c^2 S(1) + cg(2) + g(3)] + g(4)$

$= c^3 S(1) + c^2 g(2) + c \cdot g(3) + g(4)$

$\vdots$

$S(n) = c^{n-1} S(1) + c^{n-2} g(2) + c^{n-3} g(3) + \dots + c^{n-(n-1)} g(n-1) + g(n)$

$= c^{n-1} S(1) + \sum_{i=2}^{n} c^{n-i} g(i)$ $\quad n \geq 2$

Verify: by induction

Anchor: $S(2) = c^1 \cdot S(1) + g(2)$ ✓ proposed formula

$= c \cdot S(1) + g(2)$ recurrence relation

$\Rightarrow$: Assume $P(k)$: $S(k) = c^{k-1} S(1) + \sum_{i=2}^{k} c^{k-i} g(i)$

Consider $P(k+1)$: $S(k+1) = c^k S(1) + \sum_{i=2}^{k+1} c^{k+1-i} g(i)$,

+ in particular

$S(k+1) = c\, S(k) + g(k+1)$

$= c\left[c^{k-1} S(1) + \sum_{i=2}^{k} c^{k-i} g(i)\right] + g(k+1)$

$$= c^k S(1) + c \sum_{i=2}^{k} c^{k-i} g(i) + g(k+1)$$

$$= c^k S(1) + \sum_{i=2}^{k} c^{k+1-i} g(i) + g(k+1)$$

$c^{k+1-(k+1)} = 1$

$$\boxed{S(k+1) = c^k S(1) + \sum_{i=2}^{k+1} c^{k+1-i} g(i)}$$

$(P(k+1)!)$ ✓

#54 recursive defn of max, $\{a_1, \dots, a_n\}$

$a_i \in \mathbb{Z}$

Two things:

1. base case
2. recursive formula

(1) Given a set of two integers, $\{a_1, a_2\}$

max $(\{a_1, a_2\})$

if $a_1 > a_2$ then

a_1

else

a_2

fi

(2) max $(\{a_1, \dots, a_n\})$

$= \max(\max(\{a_1, \dots, a_{n-1}\}), a_n)$