Section 5.3: Decision Trees

March 19, 2008

Abstract

Decision trees are defined, and some examples given (almost every tree will be binary in what follows). Binary search trees store data conveniently for searching later. Some bounds on worst case scenarios for searching and sorting are obtained.

1 Decision Tree Definition and Terminology

Definition: a **decision tree** is a tree in which

- internal nodes represent actions,
- arcs represent outcomes of an action, and
- leaves represent final outcomes.

2 Examples of decision trees in action

A decision tree for trees! For example, the Identification of Common Trees of Iowa:
(http://www.extension.iastate.edu/Pages/tree/key.html)

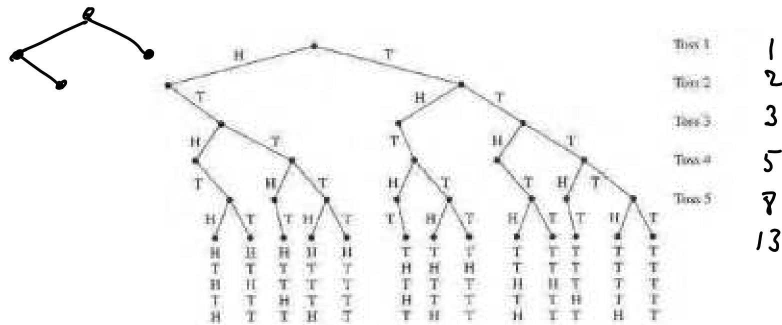


Figure 1: Figure 5.51, p. 387/452: Results of tossing a coin 5 times, no two heads in a row (binary decision tree). In the fall of 2007, a student made an interesting observation: how many leaves are there at each depth?

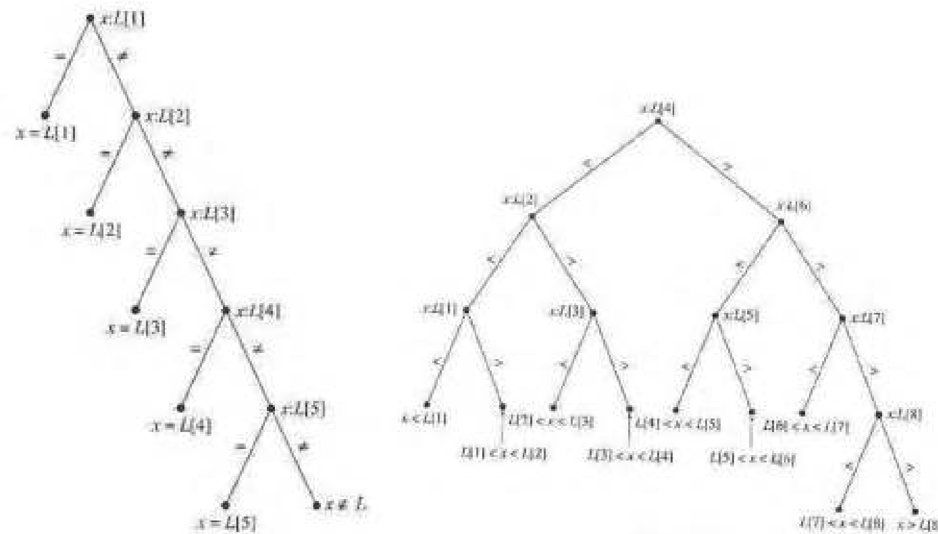


Figure 2: Figure 5.52, p. 388/453: Sequential Search on 5 elements (binary tree); Figure 5.53, p. 389/454: Binary Search on a sorted list (ternary tree, although it appears binary since those leaves corresponding to equality have been suppressed). Notice how clumsy this binary tree looks, since a power of two was used (8 elements), rather than one less than a power of two ($7=2^3-1$).

3 Lower Bounds on Searching

In particular, here are some properties about binary trees:

1 Any binary tree of depth d has at most $2^{d+1}-1$ nodes. (Proof: look at the full binary tree, Table (1), as it has the most nodes per depth.)

2 Any binary tree with m nodes has depth $d \geq \lfloor \log m \rfloor$, where $\lfloor x \rfloor$ is the **floor** function, meaning the greatest integer less than or equal to x. Again, the proof can be motivated simply by looking at the full binary tree situation: A more formal proof is by contradiction and interesting (p. 390/455):

Table 1: Adding one more node to a full binary tree bumps the depth up 1, so that if there are 2^d nodes, the depth is (at least) d. Hence, in the case of powers of 2, $d = \log m$.

Depth d	Nodes m	Nodes by depth
0	$1 = 2^1 - 1$	1
1	$3 = 2^2 - 1$	1+2
2	$7 = 2^3 - 1$	1+2+4
3	$15 = 2^4 - 1$	1+2+4+8
⋮	⋮	⋮

$n \quad 2^{n+1} - 1$

- Assume $d < \lfloor \log m \rfloor$: then $d \leq \lfloor \log m \rfloor - 1$.
- From property 1 above, $m \leq 2^{d+1} - 1 \leq 2^{\lfloor \log m \rfloor - 1 + 1} - 1 \leq 2^{\log m} - 1 = m - 1$.

Therefore, by contradiction, $d \geq \lfloor \log m \rfloor$.

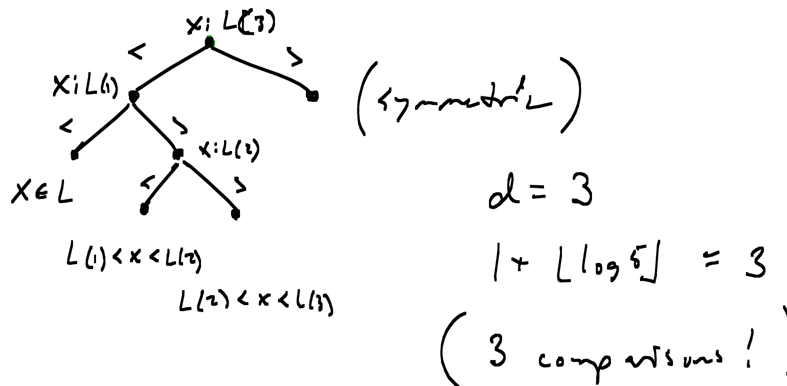
These facts lead to the following

Theorem (on the lower bound for searching): Any algorithm that solves the search problem for an m-element list by comparing the target element x to the list items must do at least $\lfloor \log m \rfloor + 1$ comparisons in the worst case.

The "+1" comes about because a decision tree representing the search problem has leaves which report the outcome of the search: hence its depth is 1 more than the depth of a tree containing only the internal nodes (representing the comparisons themselves). So, for example, at depth 0 we make the first comparison: the depth of the last internal node is actually 1 less than the number of comparisons we make, which is given by the depth of the tree (including its leaves).

If, in its worst case, an algorithm does at most this lower bound on worst case behavior is an **optimal algorithm** in its worst-case behavior. Binary search is optimal (as seen, for example, in Practice 24)

Example: : Practice #24, p. 390/454



4 Binary Search Tree

The Binary search algorithm required a sorted list; if your data is unsorted (it may be changing dynamically in time, if you're updating a database of customers, for example), you can populate a tree which approximates a sorted list, and then use a modified search algorithm (**binary tree search**) to search the list. A **binary search tree** is constructed as follows:

- The first item in the list is the root;
- Successive items are inserted by comparing them to existing nodes, from the root node: if less than a node, descend to the left child and iterate; if greater than, descend to the right child.
- If, in descending, there is no child, you create a new node.

For example,

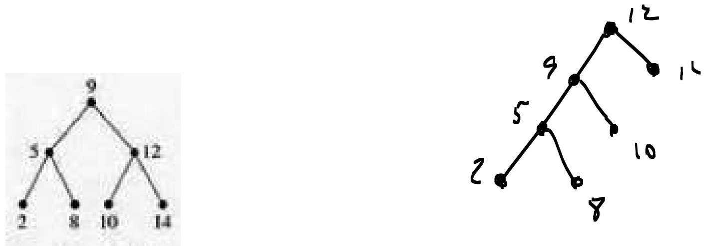


Figure 3: Figure 5.55: tree obtained from entering 9, 12, 10, 5, 8, 2, 14, in that order

Example: : Practice #25, p. 392/457. 12, 9, 14, 5, 10, 8, 2

The binary tree search algorithm works in the same way as you'd introduce a new node, only the algorithm terminates if

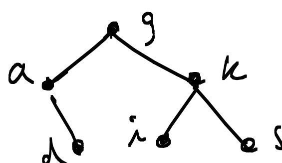
- the element is equal to a node, or
- the element is unequal to a leaf of the binary search tree.

In this case the binary search tree serves as the trunk of the decision tree for the binary tree search algorithm (minus the leaves).

Example: : Exercise #9, p. 395/459.

9a. $\lfloor \log 6 \rfloor + 1 = 3$

b. Given $\{a, d, g, i, k, s\}$



g k i s a d
(for example)

What's the worst way to enter the data into a binary search tree, if one is seeking to create a balanced tree?

5 Sorting

Examine Figure 5.56, p. 393/457:

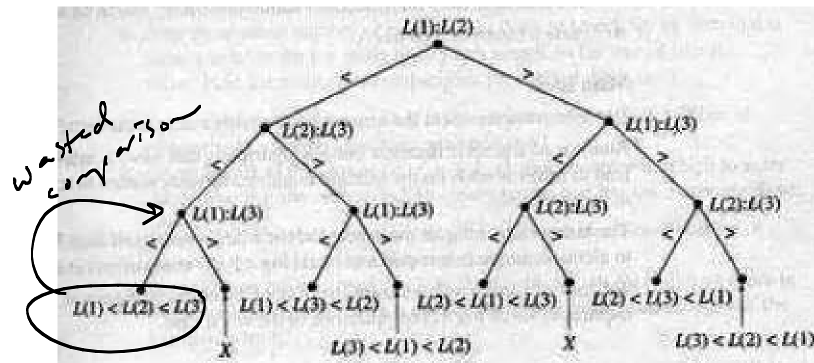


Figure 4: Figure 5.56, p. 393/457: Sorting a list (binary tree, provided distinct list elements)

In this case, we're sorting a three-element list using a decision tree. The author calls this a stupid algorithm (actually, "not particularly astute"): why?

Assuming no equal elements in the list, then this is indeed a binary (rather than ternary tree, with = included). In this case, we can also get a lower bound on sorting a list with n elements:

- There are $n!$ possible sorted lists, and there must be at least that many leaves p ($p \geq n!$). (In Figure 5.56, there are eight leaves, but only 6=3! different sorted lists).
- A worst-case final outcome in the decision tree is given by the depth d of the tree.
- Since the tree is binary, $p \leq 2^d$ (the maximum number of leaves possible at depth d).
- Taking logs (base 2), we get $\log p \leq d$, or $d \geq \lceil \log p \rceil$, where $\lceil x \rceil$ is the **ceiling** function, which yields the smallest integer greater than or equal to x.
- Hence, $d \geq \lceil \log n! \rceil$.

This is the Theorem on the lower bound for sorting: that you have to go to at least a depth of $\lceil \log n! \rceil$ in the worst case.

Exercise #23, p. 397/462, shows that this lower bound ($\lceil \log n! \rceil$) is on the order of $n \log n$ (as we discovered for mergesort).

Example: : Exercise #15, p. 395/460 .

5 coins, one lighter than the others.