# Section 7.1: Boolean Algebra Structure

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**Abstract**

First of all, note that we're only reading 7.1 through p. 468/541 (up to Isomorphic Boolean Algebras).

A Boolean algebra (named after George Boole ) is a generalization and an abstraction of the propositional logic we studied early this term, as well as the set theory which we glanced at, albeit briefly. We are really interested in using it to understand the basic elements of computer logic, however, which is based on a binary (0,1) alphabet. In this first section we are merely introduced to the fundamental concepts of Boolean Algebra.

## 1 Definition and Terminology

**Definition**: a **Boolean Algebra** is a set $B$ on which are defined two binary operations $+$ and $\cdot$, and one unary operation $'$, and in which there are two distinct elements 0 and 1 such that the following properties hold for all $x, y, z \in B$:

| | | |
|---|---|---|
| $1a.\ x + y = y + x$ | $1b.\ x \cdot y = y \cdot x$ | commutative property |
| $2a.\ (x + y) + z = x + (y + z)$ | $2b.\ (x \cdot y) \cdot z = x \cdot (y \cdot z)$ | associative property |
| $3a.\ x + (y \cdot z) = (x + y) \cdot (x + z)$ | $3b.\ x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ | distributive property |
| $4a.\ x + 0 = x$ | $4b.\ x \cdot 1 = x$ | identity property |
| $5a.\ x + x' = 1$ | $5b.\ x \cdot x' = 0$ | complement property |

The element $x'$ is called the **complement** of $x$. The algebra may be denoted $[B, +, \cdot, ', 0, 1]$.

Of these properties, certainly the distributive property 3a. may seem the strangest, since it obviously doesn't hold for the usual suspects $+$ and $\cdot$.

Notice the beautiful **symmetry** in this definition: the roles of $+$ and $\cdot$ are exactly reversed with respect to the special elements 0 and 1.

**Question**: how are these reflected in the properties of propositional logic that we studied earlier this term?

- A comment on functions $f : \{T, F\}^n \to \{T, F\}$, and example (implication).
- The difference between $\iff$ and $=$....

$\to : \{T,F\}^2 \to \{T,F\}$

| A | B | A → B |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

In Example 2, p. 465/538, the set $B = \{0, 1\}$ consisting of **only** two elements (so they must be our distinguished elements), and the binary operations of $+$ by $x+y = max(x, y)$ and that of $\cdot$ by $x \cdot y = min(x, y)$. Complements are given by $0' = 1$ and $1' = 0$. It turns out that this is another example of a Boolean Algebra.

**Example: Practice 1, p. 465/538**

$x \cdot 1 = x$

Plug in all possibilities (proof by exhaustion).

$0 \cdot 1 = 0$ ✓

$1 \cdot 1 = 1$ ✓

| · | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

| + | 0 | 1 |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 1 | 0 |

$0 + 0 = 0$

$1 + 0 = 1$

Curiously enough, $x + x = x$ in a Boolean Algebra (this is the **idempotent property**. You'll want to remember that one, for any proofs!) And since $x + x = x$, we must have $x \cdot x = x$ by the beautiful symmetry of the operations. This symmetry, known as **duality**, means that we only have to do half the work most of the time....

You may have bumped into this concept in linear algebra: for example, projection matrices are idempotent, such as

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix projects onto the first, third, and fourth dimensions; and projecting onto those dimensions a second time doesn't change anything (i.e., $A \cdot A = A$).

**Example: Practice 2, p. 467/540**

$X + X = X \rightarrow$ a. $X \vee X = X$
b. $X \cup X = X$

$X \cdot X = X \rightarrow$ a. $X \wedge X = X$
b. $X \cap X = X$

Convention $+ \rightarrow \vee$ or $\cup$

$\cdot \rightarrow \wedge$ or $\cap$

Given an element $x$ of the set $B$ of a Boolean Algebra, the complement $x'$ is the **unique** element of $B$ with the property that

$$x + x' = 1 \text{ and } x \cdot x' = 0$$

If you find $x+y=1$ and $x \cdot y = 0$, then $y = x'$.

**Example: Practice 3, p. 467/540**

Prove: $X + 1 = 1$

( Think: $X \cup S = S$ ↑ universe

or $X \vee 1 = 1$

$$\begin{aligned} X + 1 &= X + (x + x') && (5a) \\ &= (x + x) + x' && (2a) \\ &= x + x' && \text{idempotence} \\ &= 1 && (5a) \; ✓ \end{aligned}$$

Furthermore, & very cheaply, $\boxed{x \cdot 0 = 0}$

Hints for proving Boolean Algebra Equalities (p. 467/540):

- Usually the best approach is to start with the more complicated expression and try to show that it reduces to the simpler expression.
- Think of adding some form of 0 (like $x \cdot x'$) or multiplying by some form of 1 (like $x + x'$).
- Replace constants 0 & 1 using properties, e.g. 5a & 5b

- Don't forget property 3a, the distributive property of addition over multiplication, just because it seems weird!
- Remember the idempotent properties: $x + x = x$ and $x \cdot x = x$.
- It may help to frame the argument in terms of either set theory or propositional logic, to give you a framework for understanding the thrust.

**Example: Exercise 8/9, p. 475/548**

$x \cdot y' = 0 \iff x \cdot y = x$

$\Rightarrow$: Assume $x \cdot y' = 0$; show that $x \cdot y = x$

$$\begin{aligned} x \cdot y &= x \cdot y + 0 && \text{4a} \\ &= x \cdot y + x \cdot y' && \text{hyp} \\ &= x \cdot (y + y') && \text{dist.} \\ &= x \cdot 1 && \text{5a} \\ &= x && \text{4b} \end{aligned}$$

$\Leftarrow$: Given $x \cdot y = x$, show $x \cdot y' = 0$

$$\begin{aligned} x \cdot y' &= (x \cdot y) \cdot y' && \text{hyp} \\ &= x \cdot (y \cdot y') && \text{assoc.} \\ &= x \cdot 0 && \text{5b} \\ &= 0 && \text{Dual of Practice 3} \end{aligned}$$

**Example: Exercise 11a/12a, p. 476/548**

a. $x + y = 0 \Rightarrow x = 0 \wedge y = 0$

1. $x = x + 0$ — 4a
2. $x = x + (x + y)$ — hyp
3. $x = (x + x) + y$ — assoc.
4. $x = x + y$ — idempotence
5. $x = 0$ — hyp

By symmetry, $y = 0$, so $x = 0 \wedge y = 0$ QED.

b. $x = y \iff x \cdot y' + y \cdot x' = 0$

$\Rightarrow$: assume $x = y$; show $x \cdot y' + y \cdot x' = 0$

1. $x \cdot y' + y \cdot x' = y \cdot y' + y \cdot y'$ — hyp
2. $= 0 + 0$ — 5b
3. $= 0$ — 4a

$\Leftarrow$: assume $x \cdot y' + y \cdot x' = 0$; show $x = y$.
From part a), conclude that
$x \cdot y' = 0 \wedge y \cdot x' = 0$

Let's prove this by uniqueness of complements: if also $x + y' = 1$, then $y' = x'$, and $y = x$.

Consider

$$\begin{aligned}
x + y' &= x + 0 + y' && \text{(4a)} \\
&= x + y \cdot x' + y' && \text{(hyp)} \\
&= x + x' \cdot y + y' && \text{commutativity} \\
&= (x + x') \cdot (x + y) + y' && \text{dist} \\
&= 1 \cdot (x + y) + y' && \text{5a} \\
&= x + y + y' && \text{4b} \\
&= x + 1 && \text{5a} \\
&= 1 && \text{Practice 3a}
\end{aligned}$$

By the uniqueness of complements,

$x = y$ Q.E.D.

Check of the use of duality in Practice 3:

$$
\begin{aligned}
x \cdot 0 &= x \cdot (x \cdot x') && (5b)\\
&= (x \cdot x) \cdot x' && (2b)\\
&= x \cdot x' && \text{idempotence}\\
&= 0 && (5a) \ \checkmark
\end{aligned}
$$

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#4b (DeMorgan)

$$(x+y)' = x' \cdot y'$$

Show, via uniqueness of complements:

1. $(x+y) + x'y' = 1$
2. $(x+y) \cdot x'y' = 0$

1. $(x+y) + x'y' = ((x+y)+x') \cdot ((x+y)+y')$

$\ldots$

2. $(x+y) \cdot x' \cdot y' = x \cdot x' \cdot y' + y \cdot x' \cdot y'$

$= 0 + 0 = 0$

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So $x' \cdot y'$ is playing the role of $(x+y)'$, & there's only one thing that does that!

So $x' \cdot y' = (x+y)'$

We get the other piece by duality

$$x' + y' = (x \cdot y)'$$