Section 3.1: Sets

February 20, 2008

(I) What is a set? What is not a set?

loosely, a set is a bunch of objects.

We ought to be able to decide what's in a set.

Consider the set S = set of all sets not elements of themselves.

$$\left.\begin{array}{l} \text{Is } S \in S\,? \implies S \notin S \\ \phantom{\text{Is }} S \notin S\,? \implies S \in S \end{array}\right\} \text{Russell's Paradox}$$

(II) What are some important sets in mathematics?

$\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$

(III) Properties of sets?

- order

order doesn't matter

usually don't see repetition

- special sets

ϕ - empty set

$\mathcal{U}$ - universal set

(IV) How would we define concepts of sets using the nomenclature of Chapter 1?

- standard notation

$$A = \{ x \mid P(x) \}$$

x such that $\underbrace{P(x)}_{\text{is true}}$

$A = B$ - set equality means all elements are common

$(\forall x)\big((x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)\big)$

- subsets

$A \subseteq B$ - $(\forall x)(x \in A \rightarrow x \in B)$

$A \subset B$ - $(\forall x)(x \in A \rightarrow x \in B) \wedge (\exists x)(x \in B \wedge x \notin A)$

$A = B \iff A \subseteq B \wedge B \subseteq A$

- proper subset

- power set Given set A; $\mathcal{P}(A)$ = set of all subsets of A

$\emptyset \in \mathcal{P}(A)$ and $A \in \mathcal{P}(A)$

$\mathcal{P}(\emptyset) = \{\emptyset\}$

$\mathcal{P}(\{a\}) = \{\emptyset, \{a\}\}$

$\mathcal{P}(\{a_1, \ldots, a_n\})$ has 2^n elements in it.

- Cartesian product (cross product)

$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$

- union

- intersection

- complement

- set difference

(V) What properties and identities do the standard binary and unary operations on sets satisfy?

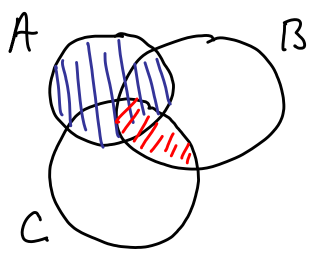
$A \cup B = B \cup A$ commutativity

$(A \cup B) \cup C = A \cup (B \cup C)$ associativity

$A \cup \emptyset = A$ "zero"

$A \cup A^c = S$ "additive inverse"

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ distributivity



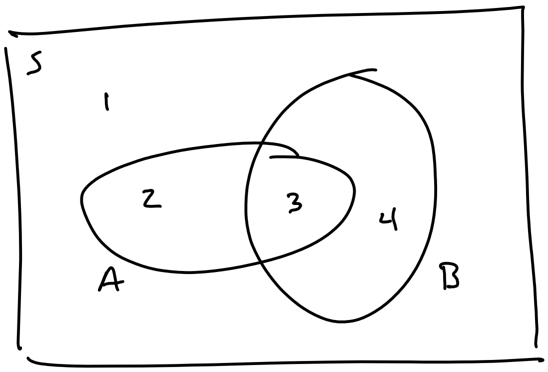
(VI) Some Surprising Properties of infinite sets

$$\begin{array}{ccc} 1 & \longleftrightarrow & 2 \\ 2 & \longleftrightarrow & 4 \\ 3 & \longleftrightarrow & 6 \\ 4 & \longleftrightarrow & 8 \\ 5 & \longleftrightarrow & 10 \\ \vdots & & \vdots \end{array}$$

Hotel: $n \to 2n$

Bus $n \to 2n-1$

Bus #1	Bus #2	Bus #3 ...
$1 \to 2^1$	$1 \to 3^1$	$1 \to 5^1$
$2 \to 2^2$	$2 \to 3^2$	$2 \to 5^2$
$3 \to 2^3$	$3 \to 3^3$	$3 \to 5^3$
$4 \to 2^4$	$4 \to 3^4$	$4 \to 5^4$
$5 \to 2^5$	$5 \to 3^5$	$5 \to 5^5$
$\vdots$	$\vdots$	$\vdots$



$S = \{1, 2, 3, 4\}$

$\emptyset = \{\}$

$A \cap B = \{3\}$

$B^c = \{2, 1\}$

To show: the power set of $\mathbb{N}$ & $\mathbb{N}$ can't be put into 1-1 correspondence.

By contradiction: assume this is a 1-1 mapping. Every subset is listed.

$\mathbb{N}$		subset of $\mathbb{N}$
1	$\longrightarrow$	S_1
2	$\longrightarrow$	S_2
3	$\longrightarrow$	S_3
4	$\longrightarrow$	S_4
$\vdots$	$\vdots$	$\vdots$

Let's construct $S \in \mathcal{P}(\mathbb{N})$ which isn't on the list.

$n \in S \iff n \notin S_n$

$(\forall i)(S \neq S_i)$

$\therefore$ S is not on the list.

Contradiction

There is no 1-1 correspondence.

#12 d, e, f

d. $1 \subseteq U$ False $(\{1\} \subseteq U$ or $1 \in U)$

e. $\{1\} \subseteq T$ True

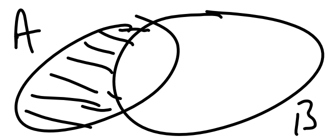
f. $\{1\} \subseteq S$ False $(\{\{1\}\} \subseteq S$ or $\{1\} \in S)$

#13 $C = \{\emptyset, \{a, \{a\}\}\}$

d. $\emptyset \subseteq C$ True

e. $\emptyset \in C$ True

#57 $(A \cup B = A - B) \to (B = \emptyset)$



#35

$$\begin{array}{c|ccc} & a_1 & \cdots & a_n \\ \hline a_1 & & & \\ \vdots & & & \\ a_n & & & \end{array}$$

$\cap : \mathcal{P}(S) \times \mathcal{P}(S) \to \mathcal{P}(S)$

$\cup :$ "

n^2 entries corresponding to (a_i, a_j)

n choices for each one.

$n^{(n^2)}$