MAT305 Midterm Exam (Fall 2009)

Name:

Directions: You must do Problem 1. You must then choose exactly six of the following eight problems (#2-9). Each solution should appear on its own separate unlined white page(s) (you may use scratch paper, however – as long as there is a clean side). They should be stapled together and submitted in order.

Besides your brain, you may use only your textbook, homeworks, and notes – nothing more in the way of reference. Do not consult with any other person; a computer; a soothsayer; a witch, elf, or leprechaun. **Good luck!**

Problem 1. (40 pts) In class I had you create a sheet that summarized Thales's contributions to mathematics. We're now going to do it again. You are to create single-sided sheets, exactly one page each, that encapsulate the mathematics of your choice of **two** of the following:

- Ancient Egyptians (pre-400 BCE)
- Ancient Babylonians (pre-400 BCE)
- The Pythagoreans

You should do these to satisfy the following conditions:

- They should be neat and clean – ready to scan (no names on these sheets, please, to preserve anonymity).
- They should fill up the entire page (I don't want to see a half page of blank space – "enlarge" your work to fill the page!).
- They should use words sparingly.
- They should incorporate all major mathematics we've discussed (and nothing beyond our textbook – i.e., do not do any web searches).
- All information on the sheet should be your own work – no images scanned and added, etc. It should all be done by your hand.
- They should be informative to any math major (not just those who've studied the history of mathematics).
- I would like them to show connections, where possible. Some ideas will be linked together; others stand on their own.

In addition, you will produce a second single sheet for each of your "works" of math history art: on this **typed** sheet you will describe your work, explaining each element that you included, and especially any subtleties. These descriptions should also be on individual sheets, clean, and anonymous.

You might begin with this "synopsis" of the history, and then "illustrate" it.

Problem 2. (10 pts) Write the base 10 numbers 574 and 475 in

a. Egyptian hieroglyphics

b. Babylonian cuneiform

c. Ionian Greek

d. Mayan numerals

e. Chinese bamboo or counting-rod numerals

f. Attic Greek numerals (see p. 19)

g. Roman numerals (see p. 20)

h. sexagesimal notation

Problem 3. (10 pts) Pythagorean triples.

a. Find the Pythagorean triples given by the Pythagorean formula

$$m^2 + \left(\frac{m^2-1}{2}\right)^2 = \left(\frac{m^2+1}{2}\right)^2$$

for which the hypotenuse does not exceed 100.

b. Prove that no isosceles right triangle exists whose three sides are integers.

c. Prove that no Pythagorean triple exists in which 1 integer is a mean proportional between the other two.

d. Find the 16 *primitive* Pythagorean triples (a, b, c) for which b is even and $c < 100$. Then show that there are exactly 100 distinct Pythagorean triples (a, b, c) with $c < 100$.

e. Show that if $(a, a+1, c)$ is a Pythagorean triple, so is $(3a + 2c + 1, 3a + 2c + 2, 4a + 3c + 2)$.

f. Prove that for any natural number $n > 2$ there exists a Pythagorean triple with a leg equal to n.

Problem 4. (10 pts) Figurate Numbers

a. List the first four hexagonal numbers.

b. Show, algebraically and geometrically, that the sum of the first n positive even integers is an oblong number

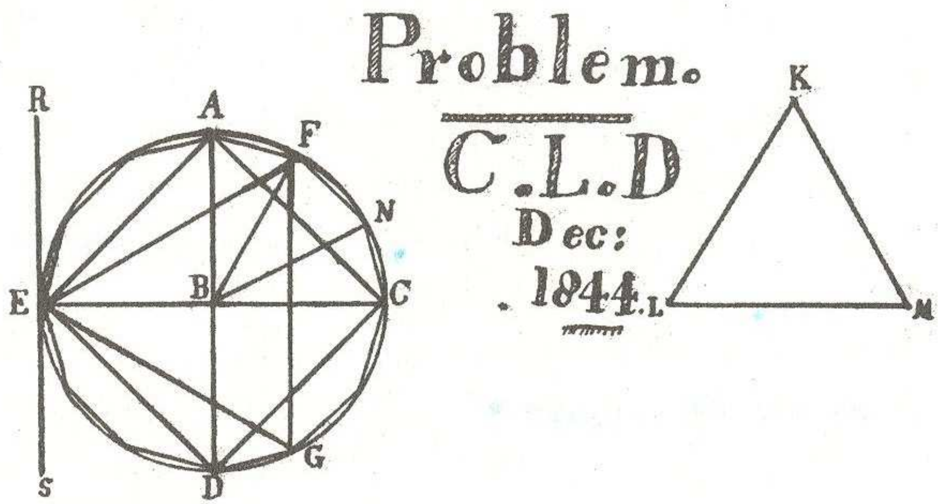
c. Prove that the sequence of m-gonal numbers is given by

$$an^2 + bn, \qquad n = 1, 2, \ldots$$

for a certain pair of rational numbers a and b.

d. Find a and b of (c) when $m = 7$.

Problem 5. (10 pts) Lewis Carroll was an angle trisector – a right-angle trisector, that is. Your job is to complete his proof. You'll need to make some deductions from his diagram, as you have only the first page!



To trisect a right angle, that is, to divide it into three equal parts.

Let there be a right angle ABC, it is required to trisect it.

Produce AB to D and make BD equal to AB, and make BC equal to AB and produce CB to E and make EB equal to BC, and join AE, ED, DC, CA. Because AB is equal to BD, and BE is common to the two triangles ABE, DBE, and the angle ABE is equal to the angle DBE, therefore the base AE is equal to the base ED; and in like manner it may be proved that all the four AE, ED, DC, CA are equal, therefore AEDC is equilateral, and because the ~~two~~ three angles of a triangle are equal to two right angles, and that the angle ABE is a right angle, (for ABC is a right angle, and EC is a straight line) therefore the angles BAE, BEA are equal to one right angle and because BA is equal to BE, therefore the angle BAE is $\frac{1}{2}$ a right angle, and in like manner it may be proved that the angle BAC is $\frac{1}{2}$ a right angle, therefore the angle BAC is a right angle, and in like manner it may be proved that the angles AED, EDC, DCA are also right angles, therefore AEDC ~~is equilateral~~ ~~that is~~ has all its angles right angles, and it was proved

A page of Charles's geometry, written when he was twelve

Problem 6. (10 pts) Algebraic geometry

a. The algebraic character of Babylonian geometry problems is illustrated by the following, found on a Strassburg tablet of about 1800 B.C. "An area A, consisting of the sum of two squares is 1000. The side of one square is 10 less than 2/3 of the side of the other square. What are the sides of the squares?" Solve this problem.

b. On a Louvre tablet of about 300 B.C. are four problems concerning rectangles of unit area and given semiperimeter. Let the sides and semiperimeter be x, y, and a. Then we have

$$xy = 1, \qquad x + y = a$$

Solve this system by eliminating y and thus obtaining a quadratic in x.

c. Solve the system of (b) using the identity

$$\left(\frac{x-y}{2}\right)^2 = \left(\frac{x+y}{2}\right)^2 - xy$$

This is essentially the method used on the Louvre tablet. It is interesting that the identity appeared contemporaneously as Proposition 5 of Book II of Euclid's elements.

Problem 7. (10 pts) Duplation and Mediation

The Egyptian process of multiplication later developed into a slightly improved method known as **duplation and mediation**, the purpose of which was mechanistically to pick out the required multiples of one of the factors that have to be added to order to give the required product. Suppose we wish to multiply 26 by 33. We may successively halve the 26 and double the 33, and so on (see the table). In the doubling column we now add those multiples of 33 corresponding to the odd numbers in the halving column. Thus, we add 66, 264, and 528 to obtain the required product, 858.

26	33	
13	66	*
6	132	
3	264	*
1	528	*
	—-	
	858	

a. Multiply 424 by 137 using duplation and mediation.

b. Prove that the duplation and mediation method of multiplication gives correct results.

c. Find, by the standard Egyptian method (discussed in class), the quotient and remainder when 1043 is divided by 28.

Problem 8. (10 pts) Regular numbers.

A number is said to be (sexagesimally) **regular** if its reciprocal has a finite sexagesimal expansion (that is, a finite expansion when expressed as a radix ("decimal", in base 10) fraction for base 60). With the exception of a single tablet in the Yale collection, all Babylonian tables of reciprocals contain only reciprocals of regular numbers. A Louvre tablet of about 300 B.C. contains a regular number of 7 sexagesimal places and its reciprocal of 17 sexagesimal places.

a. Show the necessary and sufficient conditions for n to be regular is that $n = 2^a 3^b 5^c$, where a, b, c are nonnegative integers.

b. Express, by finite sexagesimal expansions, the numbers 1/2, 1/3, 1/5, 1/15, 1/360, 1/3600.

c. Generalize (a) to numbers having base b (not the same b as in part (a)!).

d. List all the sexagesimally regular numbers less than 100, and then list all the decimally regular numbers less than 100.

e. Show that the decimal representation of 1/7 has six-place periodicity. How many places are there in the periodicity of the sexagesimal representation of 1/7?

Problem 9. (10 pts) Finger numbers were widely used for many centuries, and from this use finger processes developed for some simple computations. One of these processes, by giving the product of two numbers each between 5 and 10, served to reduce the memory work connected with the multiplication tables. For example, to multiply 7 by 9, raise 7-5=2 fingers on one hand and 9-5=4 fingers on the other hand. Now add the raised fingers, 2+4=6, for the tens digit of the product, and multiply the closed fingers, 3x1=3, for the units digit of the product, giving the result 63. This process is still used by some European peasants. Prove that the method gives correct results.