MAT360 Section Summary: 4.3
Elements of Numerical Integration (part II)

1. Summary

A few additional methods and a couple of theorems that allow us to generalize these to higher order methods.

2. **Theorems/Formulas**

Theorem 4.2: Suppose that $\sum_{i=0}^{n} w_i f(x_i)$ denotes the $(n+1)$-point closed Newton-Cotes formula, with $x_0 = a$, $x_n = b$, and $h = \frac{b-a}{n}$. Then $\exists \xi \in (a, b)$ such that

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) + \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!} \int_0^n t^2(t-1)\cdots(t-n)dt,$$

if n is even and $f \in C^{n+2}[a, b]$, and

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) + \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!} \int_0^n t(t-1)\cdots(t-n)dt$$

if n is odd and $f \in C^{n+1}[a, b]$.

One additional common rule is **Simpson's Three-Eighths rule**, which is often used if one is using Simpson's rule on an even numbered set of points (Simpson's elemental rule is defined on a panel of three points, and each additional pair gives another panel: hence, an even numbered set of points is trouble when it comes to the composite rule):

$$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] - \frac{3h^5}{80} f^{(4)}(\xi)$$

Simpson's 3/8 rule has the same order of error ($O(h^5)$), and is proportional to a fourth derivative as well (so it gets cubics right).

Open rules:

Theorem 4.3: Suppose that $\sum_{i=0}^{n} w_i f(x_i)$ denotes the $(n+1)$-point open Newton-Cotes formula, with $x_{-1} = a$, $x_{n+1} = b$, and $h = \frac{b-a}{n+2}$. Then $\exists \xi \in (a, b)$ such that

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) + \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!} \int_{-1}^{n+1} t^2(t-1)\cdots(t-n)dt,$$

if n is even and $f \in C^{n+2}[a, b]$, and

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) + \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!} \int_{-1}^{n+1} t(t-1)\cdots(t-n)dt$$

if n is odd and $f \in C^{n+1}[a,b]$.

The open rule you're already familiar with is the **Midpoint rule**:

$$\int_{x_{-1}}^{x_1} f(x)dx = 2hf(x_0) + \frac{h^3}{3}f''(\xi)$$

Open rules are good for working with some improper definite integrals, such as

$$\int_0^1 \frac{dx}{x^{\frac{1}{2}}}$$

because we can't evaluate the integrand at the endpoint $x = 0$.